Mathematics for Industrial Technicians

CHESTER PACHUCKI

Loop College
Chicago

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To

CUTIE

Constance, Carl, Charissa, and Charla

Contents

UNIT II ESSENTIALS OF ALGEBRA

UNIT III ADVANCED TOPICS

Preface

Rapid technological changes have led to the development of educational programs designed to provide scientists and engineers with support personnel. New applications of scientific principles have produced an automated-computerized industrial complex. To keep pace with these advancements, the engineering-scientific-production labor force has discovered an urgent need for another important member of the team, *the technician*. Presently, there is a recognized shortage of qualified technicians.

Technician-type programs have developed exponentially since the advent of Title VIII of the National Defense Education Act of 1958. Most of the expansion in this field has been assumed by the junior colleges, community colleges, and the technical institutes of the nation. Programs have varied in breadth, depth, scope, and philosophy. There are even different classifications for a technician depending on whether he functions in engineering, industrial production, research, sales, service, or medicine. There is general agreement, however, that a technician is a highly specialized person whose position relates to a wide occupational range bounded by the engineer-scientist on the one extreme and the craftsman on the other. He is a worker who has acquired an educational background that qualifies him to adopt or incorporate a process, technique, design, or the materials of production associated with an engineering-scientific development. A technician's immediate concern is to apply rather than to acquire or discover new scientific principles.

It is within this context that *Mathematics for Industrial Technicians* is written. This book is designed to provide post-high school students with those mathematical skills that are requisite to the complementary science, engineering, and specialized courses that make up the technician's curriculum. Topics are presented in a progressive sequence to fit the needs and abilities of the technician. Throughout the text the author has attempted to create interest in mathematics as it pertains to the field of specialization. Such fundamental engineering-scientific principles as density, specific gravity,

simple stress, resolution of force systems, ac-dc circuits, and units of conversion are defined and then incorporated with appropriate mathematical concepts as the basis for various exercises

This book is not intended for the traditional college mathematics courses that reflect, to some degree, theoretical and abstract approaches Primarily, emphasis is directed toward mathematical competence associated with job-entry responsibilities of the technical worker, stressing such relationships as, $°F = \frac{9}{5}°C + 32°$, rather than the unknown counterpart, $y = mx + 3b$ Furthermore, *Mathematics for Industrial Technicians* is designed to meet the mathematics requirements suggested in curriculum guides published by the various professional and governmental affiliates involved with this new development in higher education

Specifically, this book is organized about the following three distinct areas of mathematics as they relate to the technician

I. Fundamentals of Mathematics

Review of the basic arithmetic operations involving fractions, decimals, and percentages, application of significant figures, scientific notation and measurements, introduction to dimensional analysis, use of the slide rule and ratios-proportions and variations This section lends itself to the Allied Health Field and could be adopted for use in a course such as Chemical Calculations

II. Essentials of Algebra

Comprehensive treatment of the processes of algebra developed around signed numbers, equations, roots and radicals, fractions, quadratics, factoring, and plotting functions

III. Advanced Topics

Intensive survey of geometry, introduction to trigonometry, vectors, complex numbers, arc measurements, and logarithms, along with several pertinent concepts from analytic geometry

Mathematics for Industrial Technicians can be utilized most effectively as an instructional module that will keep the curriculum current and realistic

Chester Pachucki

Mathematics for Industrial Technicians

UNIT I

Fundamentals of Mathematics

This unit is developed around those processes considered introductory to all branches of mathematics. Most scientific-engineering principles associated with mathematics are resolved, eventually, by the fundamental operations of arithmetic. These basic concepts are further extended to include exponents, dimensional analysis, ratios, proportions, and variations along with the use of the slide rule. Emphasis is also directed toward units of measurements and their conversion from the English System to the Metric System. Study in this area is considered pertinent for students who will be entering the fields of chemistry, nursing, and other health related fields.

Dimensional analysis, a unique feature of Unit I, is a technique frequently employed in the derivation of physical relationships called formulas. This topic also concerns itself with the behavior of units (measurement) of the various elements of a formula during that operation referred to as solving an equation. The involvement of formulas is intended, primarily, as an introduction to some of the relevant engineering-scientific principles that may eventually become the concern of the technician.

CHAPTER 1

General Concepts

Mathematics, more than any other area of knowledge, is the predominant factor in the development of the atomic-automated-computerized world we live in today. Useful applications of long-established scientific principles are realized, by and large, because we have increased our ability to transmit these phenomena into meaningful mathematical relationships.

The growth and status of the technician, likewise, will depend a great deal on the individual's capacity to "handle" the tools of mathematics with respect to his particular area of specialization. An electronic technician will apply vector algebra to facilitate an a-c circuit analysis. The draftsman will rely more on his background in geometry and trigonometry than on the drawing techniques, when detailing a layout or a design. A chemical technician must be prepared to balance chemical equations and determine concentrations of particular solutions. The structural technician, at times, may be required to construct a force polygon as a further basis for the study of structural equilibrium.

These are only limited examples of how mathematics relates to the work of the technician. Presently, our concern will be with the development of those mathematical skills that will lead to a productive entry into the labor force. The development will begin with **numbers** and it should be mentioned that this facet of mathematics has had a most interesting, though somewhat tedious, process of evolution.

1-1 NUMBERS AND NUMERALS

There are several engineering-scientific terms or concepts that are used interchangeably by laymen such as stress-strain, force-pressure, screws-bolts, heat-temperature, mass-weight, and **numbers-numerals.** Referring to numbers-numerals, in strict mathematical language the *symbol VII* is the *Roman numeral* representing the *number* 7. Similarly, the *symbol* 4 is the *Arabic numeral* representing the *number* 4. After recognizing this distinction between a

numeral (symbol) and a number (name), we shall allow ourselves, as technicians, the fringe benefit of referring freely to the numerical symbols, 1, 2, 3, 4, 5, , as numbers

1-2 NATURAL NUMBERS

The number system with which the technician will be working can be represented geometrically as corresponding points on a line Let $X X$ designate a straight line that can be extended, without bounds, in either direction (Fig 1-1) On this line an arbitrary *point* 0 is selected, initially, to designate the *starting point, the origin* or the *zero point* From the origin 0, moving to the right, successive equal lengths are marked off and identified by the points $P_1, P_2, P_3, P_4,$, corresponding, respectively, to the numbers 1, 2, 3, 4,

The *number* 1 in the *notation* P_1 is *called a subscript* and is used to define, in the illustration a particular point P along the line $X'X$ Likewise, the number 2 in the notation P_2 is also a subscript and defines another distance OP_2 on the same line Subscripts are used to identify the uniqueness of several related quantities that appear in a particular discussion This identification holds throughout the given discussion Subscripts carry no arithmetic value since they are merely designations The *symbol* ′ in $X X$ is called *prime*, the *notation* $X'X$ *reads x prime x*

In mathematics three dots, , are used to indicate that a certain pattern has been established and continues with each successive term bearing this orderly relationship For example, the sequence, 2, 4, 6, 8, 10, 12, 14, 16 may be written as 2, 4, 6, , 16

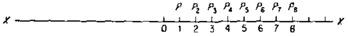


Figure 1-1

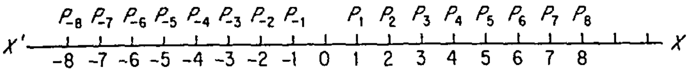
The points $P_1, P_2, P_3, P_4,$, corresponding to the numbers 1, 2, 3, 4, , establish the *scale of natural numbers* or *positive integers*, also referred to as *whole numbers*

1-3 NEGATIVE INTEGERS

Since the line $X'X$ extends in both directions, points and numbers can also be identified to the left of the origin, or zero Although the successive lengths on the left will be measured off equal to the respective markings on the right, the change in direction must be reflected somehow in the notation of the points (Fig 1-2)

In the fields of science, engineering, and mathematics, some laws and principles take into consideration the element of direction, such as an applied force, the flow of current, and a rotating disc If a force acting in one direction is taken as positive (+), then a force acting in opposition, a counterforce, is considered negative (−) for purposes of computation If a counterclockwise motion is taken as positive, clockwise would be taken to mean

Figure 1-2



negative. Thus, if the integers to the right of zero are termed positive, then the integers to the left are defined as negative.

The subscripts, to maintain an orderly number system, would then be designated with a minus (−) sign such that the points $P_{-1}, P_{-2}, P_{-3}, P_{-4}, \ldots$, correspond to the negative integers, $-1, -2, -3, -4, \ldots$.

In mathematics, numbers are assumed positive (+) unless otherwise noted. Thus, +2 and 2 are both considered positive and represent the same quantity.

1-4 REAL NUMBERS

The need for counting, to keep track of possessions, initially led to the development of whole numbers or integers. Losses and reversals of direction in laying out plots of land may have suggested the concept of negative integers or other symbols. Whole numbers eventually proved inadequate and the concept of half or part emerged in some form of notation resembling a fraction.

Natural numbers and negative numbers are only the foundation of a more comprehensive number system. This system includes **rational numbers** and **irrational numbers.**

Rational numbers are integers and all other numbers that can be expressed as the quotient of two integers; $\frac{12}{61}, \frac{2}{3}, \frac{30}{13}, \frac{-7}{9}, \frac{5}{4}, \frac{138}{-17}, \ldots$. These *numbers* are also *called fractions.* In general form; p/q is referred to as a *rational number*, where p and q are both integers and $q \neq 0$. *The symbol,* $\neq$, *is read; q is different from zero, other than zero, or q is not equal to zero.*

Included along with integers and fractions are **decimals.** *Decimals are fractions that have a denominator of* 10 *or some multiple of* 10; $2.0 = \frac{20}{10}$, $0.20 = \frac{20}{100}$, $5.25 = \frac{525}{100}, \ldots$.

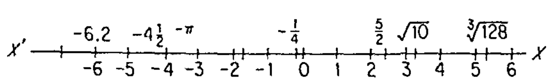
Numbers that cannot be represented as the quotient of two integers are called irrational numbers. Possibly the most familiar irrational number is the quantity that expresses the ratio of the circumference of a circle to its diameter, π, (pi), where $\pi = 3.1416$ (approximately). Other examples of irrational numbers are; $\sqrt{2}$, e, $\sqrt{3}$, $\sqrt[3]{39}, \ldots$.

Although irrational numbers are usually expressed as decimals, it does not follow that all decimals are irrational.

The real number system includes 0 *and all numbers, both positive and negative, that are classified as rational and irrational.* This system can be represented by a number scale (Fig. 1-3).

There are other numbers that the technician will be working with called **imaginary numbers** and **complex numbers.** Presently, the discussions will involve, primarily, the **real number system.**

Figure 1-3



In developing the number scale (Fig. 1-2), a vertical line could have been used in place of the conventional horizontal line. The resulting discussion would not have changed except that the integers above the origin or zero would then be called positive and the integers below the origin, negative (Fig. 1-4).

If the vertical scale (Fig. 1-4) were now to be superimposed on the horizontal scale (Fig. 1-2) a very useful and important mathematical concept would be developed, referred to as a **system of coordinates** (Fig. 1-5).

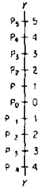


Figure 1-4

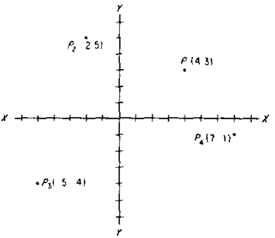


Figure 1-5

The intersection of the line X′X and line Y′Y at an angle of 90° (*perpendicular*) *represents the rectangular coordinate system, where the point of intersection is called the origin.* The two given lines, $X'X$ and $Y'Y$, are called the *axes*.

The coordinate system provides a convenient method of representing important mathematical and engineering data, graphically or geometrically. It offers the technician an opportunity for a "comprehensive look" at the relevance of the various factors involved.

1-6 REFERENCE POINTS

One way to define or *plot* a point, in a plane, is to give its direction and distance from the axis. In Fig. 1-5, the point P_1 is located three units above $X'X$ and four units to the right of $Y'Y$. P_2 locates a point two units to the left of the $Y'Y$-axis and five units above the $X'X$-axis. Similarly, P_3 locates a point four units below $X'X$ and five units to the left of $Y'Y$. Furthermore, P_4 is plotted one unit below $X'X$ and seven units to the right of $Y'Y$.

The units or numbers that define a point such as (7, −1), *are called coordinates. The first number* (7) *is referred to as the abscissa of the point and the second number* (−1) *is called the ordinate of the point. The abscissa is plotted*

with reference to the X'X-axis, or x-axis, and the ordinate is plotted with reference to the Y'Y-axis or y-axis.

Technicians will be involved continually with the concept of an origin or reference point from which other meaningful measurements are taken.

In mechanical technology the point of reference of a layout may be a reamed hole or a gear center (Fig. 1-6). The point of reference for an electronic technician may be the terminals of a power supply, whereas the chemical technician may refer back to standard temperature-pressure conditions. The civil technician, on the other hand, in making a traverse (survey) may tie in to a well-known and established landmark, usually a bench mark.

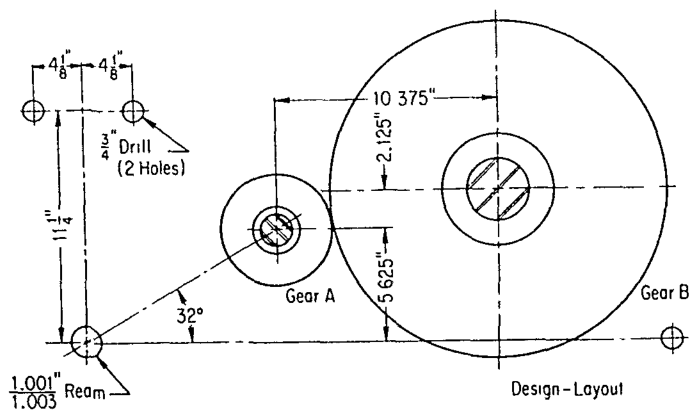


Figure 1-6

1-7 MEASUREMENT

To convey intelligence, numbers must be associated with other properties, such as units of measurement. A brief study of the design layout, Fig. 1-6, will bear this out. Numbers and properly selected units alone, however, will not guarantee that the end result is *reliable*. This leads to an important consideration, one that involves *precision* and *accuracy of measurement*.

There is probably little doubt about the scientific interpretation of a number when it refers to a quantity such as 3, 9 volt Type M, transistor radio batteries. It's inconceivable, assuming proper manufacturer's identification, that the value 3 could possibly become an item to challenge. On the other hand, the power rating brings in an element that involves measurement.

Measurement relies primarily on precision of comparison to a known standard. At the factory, this probably involves the instruments of quality control. In the laboratory, however, this might be accomplished, in this instance, by means of a volt-ohm meter. Reading a meter is a subjective judgment based on several variables, foremost of which is the experience of the technician. The precision of the instrument and the accuracy of measurement are also factors that must be taken into account.

In the process of completing the layout of Fig. 1-6, the draftsman used angular units and linear units along with fractions and decimals to locate

certain centers Depending on local engineering office standards, these dimensions carry various tolerances *Fractional dimensions* may carry a *tolerance* of $\pm\frac{1}{64}$, which means plus or minus $\frac{1}{64}$ of an inch *Tolerance refers to the variation from the nominal dimension* Decimal figures carry a tolerance of $\pm 0\ 005$ in This means that the dimension that locates the center line of the $\frac{3}{4}$ in holes can be within the limits of $11\frac{15}{64}$ in and $11\frac{17}{64}$ in , where $11\frac{1}{4}$ is considered the nominal dimension Furthermore, the dimensions locating the center of gear *B* can vary between 10 370 in and 10 380 in (horizontally) and 2 120 in and 2 130 in (vertically) and still meet engineering requirements or standards

In locating the various centers for the design, the craftsman will be required to use several different instruments, among which will be a machinist's scale, precision within $\frac{1}{64}$ of an in , a protractor with vernier adjustment, precision within 0 1 of a degree, and a height gauge along with other tools having micrometer precision, 0 001 in

Thus it becomes apparent that a figure (number or dimension) reflects several considerations These considerations appear later as **significant figures** in the final form of the number

1-8 SIGNIFICANT FIGURES

Significant figures refer to those digits of a number that are considered reliable The **reliability** of a number is based on **precision** and **accuracy**. *Precision relates to the sensitivity of the measuring instrument whereas accuracy refers to the technique of interpreting the reading*

An important consideration confronting the technician in the solution of problems is the reliability of the answer in terms of the given data that may come from different sources, such as mathematical tables, manufacturers specifications, professional handbooks, field notes, drawings, laboratory calculations, and other sources affiliated with the particular area of specialization The **final computations** can only **reflect** the **reliability** of the **crudest measurement**

All the digits expressed in a number that represent an engineering-scientific measurement or some quantitative value are considered significant figures The number, 3 14 has **three significant figures, 3, 1, 4** Likewise, the following numbers have the same three significant figures, **0 314, 31.4, 0 0314, 314**, and **0 00314** *The position of the decimal point does not determine the number of significant figures* Several other examples will follow

72 15 is reliable to **four**	significant figures, 7, 2, 1, 5
72,150 is reliable to **five**	significant figures, 7, 2, 1, 5, 0
0 07215	**four** significant figures, 7, 2, 1, 5
1 07215	**six** significant figures, 1, 0, 7, 2, 1, 5
7,215 0	**five** significant figures, 7, 2, 1, 5, 0
721,500	**six** significant figures, 7, 2, 1, 5, 0, 0
0 00042	**two** significant figures, 4, 2

When zeros appear as final digits after the decimal point, they are considered significant only if they are so specified.

If the zeros appear after a decimal point, but *preceding* other *digits*, the *zeros* are *not significant*. This happens in those cases in which the numbers are less than 1, such as 0.07215 and 0.00042. In either situation these zeros are not considered significant figures. Furthermore, if the zero appears before the decimal point, known factors and usage defining the number will determine whether or not the zeros are significant. This condition involves numbers greater than 1, as, for example, 72,150. and 721,500. Here, the zeros are considered significant.

A clear distinction should be made among reliability, precision, and accuracy. It does not follow that a number with three significant figures is representive of a more accurate measurement than a number with one significant figure.

20.0 has three significant figures with accuracy to 0.1
0.20 has two significant figures with accuracy to 0.01
0.002 has one significant figure with accuracy to 0.001

All of these numbers must be considered reliable; the significant figures reflect accuracy of measurement within the precision limits of the instrument.

In an arithmetic operation a number may emerge containing more (or less) significant figures than the given data. This condition is resolved by applying the principle known as **rounding off numbers.**

1-9 ROUNDING OFF NUMBERS

Rounding off numbers refers to that process whereby the results of various mathematical operations are recorded in terms of the reliability of the given data. The practice involving rounding off numbers may vary among offices or industries; however, the following rules apply most often.

1. If the number being dropped is less than 5, the last digit is left unchanged.
   3.141 rounded to 3 figures would be recorded as 3.14
2. If the number to be dropped is 5 or more, the last remaining digit is increased by 1.
   3.1416 rounded to 4 figures would be written as 3.142
3. Only the digit immediately to the right of the number being dropped is considered in the process of rounding off.
   6.348 rounded to 3 digits becomes 6.35, whereas
   6.348 rounded to 2 digits becomes 6.3.
4. During an intermediate step it is permissible to retain accuracy one place beyond the crudest measurement.
5. Tabular values should be used as they appear in context.
6. No answer should reflect accuracy beyond the crudest measurement.

EXAMPLE 1-A

Find the value of 3 14 × 4 2/6 0 and round off to the proper number of significant figures

Solution

Multiply 3 14 by 4 2 and divide the product by 6 0 Thus, 3 14 × 4 2 = 13 188, which should be rounded off to 1 digit beyond crudest measurement, or 13 19

Next, divide 13 19 by 6 0

13 19/6 0 = 2 198, rounded off to 2 2 to conform to the accuracy of the crudest measurement (6 0 and 4 2)

EXAMPLE 1-B

Add the following numbers and round off the sum to the proper number of significant figures

$$
\begin{array}{r}
21\ 761 \\
5\ 20 \\
13\ 1 \\
0\ 93 \\ \hline
40\ 991
\end{array}
$$

This sum must be rounded off to one decimal place in conformity with the given data, where 13 1 is the least precise measurement

Thus, 40 991 is rounded off to 41 0

EXAMPLE 1-C

Find the density of a substance with a mass, m, of 13 36 grams (g) and a volume, v, of 3 2 cubic centimeters (cm^3)

Solution

$$\text{Density} = \frac{\text{mass}}{\text{volume}}, \text{ or } D = \frac{m}{v}$$

Thus, $D = \frac{13\ 36\ \text{g}}{3\ 2\ \text{cm}^3}$ To insure accuracy to one decimal place, the arithmetic division should be carried out to two decimal places, and the quotient should be rounded off

$$
\begin{array}{r}
4\,1\,7 \\
3\,2\,\overline{)\,1\,3\,3\,6\,0} \\
1\,2\,8 \\ \hline
5\,6 \\
3\,2 \\ \hline
2\,4\,0 \\
2\,2\,4 \\ \hline
1\,6
\end{array}
\qquad \text{rounded off to } 4\ 2
$$

Hence, $D = 4\ 2\ \text{g/cm}^3$

1. Lay out a rectangular coordinate system, similar to Fig. 1-5, and plot (graph) the following coordinates.

a. $P(-3, 4)$	e. $P(4, 3)$	i. $P(0, 5)$
b. $P(3, 4)$	f. $P(-4, 3)$	j. $P(3, -4)$
c. $P(4, -3)$	g. $P(-3, -4)$	k. $P(-5, 0)$
d. $P(-4, -3)$	h. $P(5, 0)$	l. $P(0, -5)$

Connect all of these points with a smooth curve. What geometric form seems to emerge.

2. Lay out the points;

a. $(\pi, -\pi)$ b. $(0, 0)$ c. $(-\pi, \pi)$

Connect the three points and identify the geometric figure.

3. Classify the following numbers as (a) rational, or (b) irrational.

$$36, -36, 6, \frac{36}{6}, \frac{9}{5}, \frac{13}{7}, \frac{-8}{2}, \sqrt{6}, -\sqrt{4}, 2\pi, \sqrt{\pi}, \frac{999}{1{,}000},$$
$$\frac{1{,}000}{999}, 3.01, 72.624, \frac{3.6}{3}, 3\frac{1}{3}, 5\frac{2}{5}, 7\frac{3}{4}, 21\frac{9}{10}, 2.5,$$

4. State briefly the distinction between a digit and an integer, if there is a distinction.

5. Give several examples of reference points as they may apply to business and industry.

6. Indicate the amount of significant figures in each of the following numbers.

a. 3.21	e. 0.00321	i. 5.6720
b. 032.1	f. 0.003210	j. 0.05002
c. 3210	g. 36,511.	k. 0.0050
d. 0.00321	h. 90.20	l. 0.000100

7. Round off the given numbers to three decimal places.

a. 21.89992	d. 0.0444	g. 1.05547
b. 21.89949	e. 1.06456	h. 1.0001
c. 456.233338	f. 2.40	i. 345.6

8. Perform the indicated arithmetic operations and express answers with proper significant figures.

a. Add	b. Add	c. Add
36.32	126.2	26.365
9.07	3.65	10.20
17.056	20.01	18.0
6.1	0.009	10.4

d. Divide 13.510 by 2.25

e. Divide 9.456 by 4.51

f. $\dfrac{3.14 \times 6.75 \times 13.1}{26.2 \times 1.57} =$

9. Find the number of gallons in a cylindrical tank whose diameter, d, is 18.2 in. and whose height, h, is 44.6 in.

$$\text{Volume, } V = \pi\frac{d^2}{4} \times h, \ \pi = 3.1416$$

$$231 \text{ cubic inches (in.}^3) = 1 \text{ gallon}$$

10. Complete the number scale in Fig. 1-7 by locating the intermediate points (integers only).

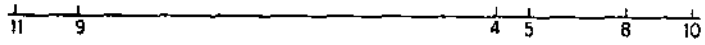


Figure 1-7

CHAPTER 2

Common Fractions

The use of electrical calculators has provided the technician with a speed-up method of by-passing the rigors of "long division." By no means, however, has the development of electronic equipment in any way eliminated the need for fractions, decimals, and percentages. Most of the mathematics presently used in industry relies basically on the manipulation of quantities reduced down to this form.

2-1 DEFINITIONS

In Chapter 1 we referred to the quotient of two integers as a rational number, and in the language of fractions this same quotient is called a *common fraction*. Thus, $\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \frac{37}{49}, \frac{49}{37}, \ldots$, represent fractions. The whole number (integer) above the line of division is called the *numerator* and the number below the line is called the *denominator*.

Decimals and percentages can be viewed as special cases of fractions. The denominator of a decimal fraction is expressed as a multiple of 10. Percentage means per hundred, which implies a denominator of 100.

A familiar example of common fractions can be illustrated with the architect's scale (Fig. 2-1).

The top scale represents an inch divided into sixteen equal parts or denominations (denominators) of sixteenths, $\frac{1}{16}$. The number of times this

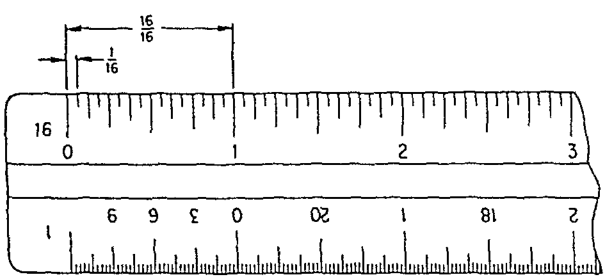


Figure 2-1

denomination is taken or counted appears as the numerator, three-sixteenths, $\frac{3}{16}$, nine-sixteenths, $\frac{9}{16}$, etc Notice also that sixteen-sixteenths represents 1 in, or $\frac{16}{16} = 1$ This last expression is an example of the *fundamental identity of multiplication*

$$\frac{a}{a} = 1, \text{ where } a \times 1 = a \text{ if}$$

$$a \neq 0$$

If the denominator of a fraction is equal to zero, the quotient is said to be indeterminate This means that the numerical value of p/q cannot be determined arithmetically if $q = 0$

$$\text{If } \frac{p}{q} = r,\ p \neq 0 \text{ and } q \neq 0, \text{ it follows that}$$

$$p = q \times r, \text{ however, } \frac{p}{q} \neq r \text{ if } q = 0$$

since there exists no value of r that satisfies the condition $0 \times r = p$, when $p \neq 0$

$$\frac{12}{3} = 4 \text{ and } 12 = 4 \times 3, \text{ but}$$

$$\frac{12}{0} \neq r, \text{ since } 12 \neq 0 \times r$$

If the numerator of a common fraction is smaller than the denominator, the fraction is called a proper fraction

$$\frac{1}{2}, \frac{3}{5}, \frac{13}{16}, \frac{63}{64},$$

If the numerator of a common fraction is larger than the denominator, the fraction is called an improper fraction

$$\frac{5}{3}, \frac{16}{13}, \frac{64}{63},$$

Whenever a *whole number* or *integer* is *combined* with a *fraction*, such as in the linear measurement $1\frac{5}{16}$ in, the quantity $1\frac{5}{16}$ is termed a *mixed number* Actually, the mixed number is an expression of the sum of a whole number and a fraction

$$1 \text{ in} + \frac{5}{16} \text{ in} = 1\frac{5}{16} \text{ in or}$$

$$1 + \frac{5}{16} = 1\frac{5}{16}$$

The mixed number also can be expressed as an improper fraction:

$$\frac{16}{16}\text{ in.} + \frac{5}{16}\text{ in.} = \frac{21}{16}\text{ in., where}$$

$1\frac{5}{16}$ and $\frac{21}{16}$ are considered *equivalent expressions.*

Some of the arithmetic operations involving fractions are closely associated with the concept of **factors**, usually **prime factors.**

A factor is defined as a multiple of a mathematical expression (usually a product).

$3 \times 8 = 24$, 3 and 8 are called **factors** of 24. Furthermore, 3 is a *prime number*, therefore, *a prime factor* of 24.

A prime number is an integer or whole number larger than 1 whose only factors are the number itself and 1.

It follows that 8 is not a prime number because $8 = 2 \times 2 \times 2$. Thus, $24 = 2 \times 2 \times 2 \times 3$, where the product now consists only of prime factors.

Another method of writing the cumbersome product $2 \times 2 \times 2 \times 3$ is to enclose each factor within parentheses ( ). Thus, $2 \times 2 \times 2 \times 3$ can be expressed as $(2)(2)(2)(3) = 24$, where $(2)(2)$ is another way of representing the product 2×2, etc.

The product $(2)(2)(2)(2)$ can be further simplified by introducing the concept of **exponents.** *An exponent indicates the number of times a factor appears in a given expression:*

$$(2)(2)(2)(2) = (2)^4\text{. The exponent in this}$$

illustration is 4. Similarly,

$$(5)(5) = (5)^2\text{, exponent, 2}$$

$$(-3)(-3)(-3) = (-3)^3\text{, exponent, 3}$$

$$(a)(a)(a)(a)(a) = (a)^5\text{, exponent, 5}$$

$$(b) = (b)^1\text{, exponent, 1, which is seldom indicated.}$$

Several prime numbers are: 2, 3, 5, 7, 9, 11, 13, 17, 19, 23, 29, . . . , 113, 127, 131, . . . , 709, 719, 727, . . . , 907, 911, 919, . . . , 2,161, 2,179, 2,203, ...

2-2 FUNDAMENTAL PRINCIPLES OF FRACTIONS

Several principles are considered fundamental to the mathematical operation involving fractions. These will be stated as rules, followed by a simple demonstration.

1. *The value of a fraction remains unchanged when both numerator and denominator are multiplied by the same number or factor where the number or factor is other than zero.*

$$\frac{5}{16} = \frac{5(2)}{16(2)} = \frac{10}{32}$$

$$\frac{3}{10} = \frac{3(5)}{10(5)} = \frac{15}{50}$$

In general form.

$$\frac{a}{b} = \frac{a(c)}{b(c)} = \frac{ac}{bc}, \text{ where } b \neq 0 \text{ and } c \neq 0$$

$\frac{a}{b}$ and $\frac{a(c)}{b(c)}$ are called **equivalent fractions**

The two examples can be verified with the use of an architect's scale and an engineer's scale (Fig 2-2)

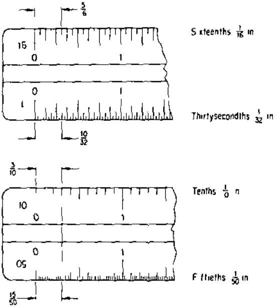


Figure 2-2

2 *The value of a fraction remains unchanged when both numerator and denominator are divided by the same number if the number is other than zero*

$$\frac{36}{64} = \frac{36 - 4}{64 - 4}, \text{ or } \frac{\frac{36}{4}}{\frac{64}{4}} = \frac{9}{16}$$

$$\frac{21}{30} = \frac{21 - 3}{30 - 3} = \frac{\frac{21}{3}}{\frac{30}{3}} = \frac{7}{10}$$

In general form.

$$\frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}}, \text{ where } b \neq 0 \text{ and } c \neq 0$$

$$\frac{a}{b} \text{ and } \frac{\frac{a}{c}}{\frac{b}{c}} \text{ are equivalent fractions.}$$

Again, with the use of scales (Fig. 2-3), the two examples can be verified.

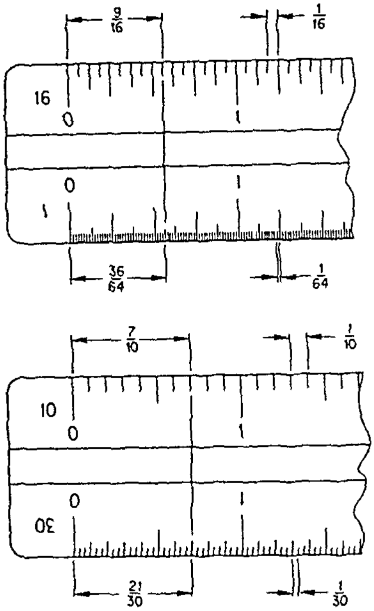


Figure 2-3

2-3 REDUCING FRACTIONS

The fundamental principles of fractions provide the technician with a technique that enables him to simplify cumbersome expressions into more manageable equivalent fractions. It is highly desirable, in some arithmetic operations, to combine fractions after they have been reduced to lowest terms. A **fraction** is considered to be in **lowest terms** when there is **no** other **factor** that will **divide** into both **numerator** and **denominator without remainder.**

States as a principle: *To reduce a fraction to its lowest terms, divide both numerator and denominator by the highest common factor. This process is also referred to as cancellation of like factors.*

There is no specific rule, in itself, that leads directly and instantaneously to locating the highest common factor for every condition. There are a few generalizations, however, which might prove worthy of exploration.

If a factor is not suggested by initial inspection, then the numerator may be studied as a possible factor of the denominator *or the denominator* as a possible factor of the numerator, depending on which is smaller.

EXAMPLE 2-A:

Simplify the fraction $\frac{17}{153}$

Solution

Since the numerator, 17, is a prime number, this fraction can only be reduced if the denominator contains 17 as a factor If 17 divides into 153 without remainder, it is a factor

$$17\overline{)153}\quad 9$$
$$\underline{153}$$

Thus the factors of 153 are 9 and 17 Therefore, $17/153 = 17/(17)(9) = (17)(1)/(17)(9) = 1/9$, where $17/17 = 1$, usually indicated by canceling

$$\frac{17}{153} = \frac{\overset{1}{\cancel{(17)}}}{\cancel{(17)}\,(9)} = \frac{1}{9}$$

Another technique suggests starting with prime numbers as possible factors This method may not provide all the multiples immediately, but it could lead possibly to a reduced expression, which may suggest other factors

$$175 = 5(35) = (5)\,(5)\,(7), \text{ or } (7)\,(5)^2$$

There are those situations in which a fraction may be more meaningful if it is not reduced to its lowest terms For example, a surveyor's tape is marked off in tenths of a foot, $\frac{1}{10}$ ft, thus a dimension that carries with it a fractional part of a foot, such as $\frac{4}{10}$ ft, would not be reduced to $\frac{2}{5}$ ft Reducing here would break with consistency of field measurement

This illustration was introduced to re-emphasize that the use of numbers will be governed by how they are obtained and applied

EXERCISES 2-1

Reduce the following fractions to lowest terms

1. $\frac{9}{21}$	*2.* $\frac{81}{27}$	*3.* $\frac{105}{515}$
4. $\frac{119}{153}$	*5.* $\frac{261}{29}$	*6.* $\frac{0}{73}$
7. $\frac{220}{330}$	*8.* $\frac{623}{979}$	*9.* $\frac{423}{63}$
10. $\frac{509}{701}$	*11.* $\frac{331}{991}$	*12.* $\frac{16a}{48a}$
13. $\frac{72(a)\,(c)}{8(b)\,(c)}$	*14.* $\frac{114b^2}{285b}$	*15.* $\frac{111a^3}{185a^2}$

2-4 ADDITION AND SUBTRACTION OF FRACTIONS

When fractions enter into a mathematical consideration, they will usually be bound by a common unit of measurement for a given condition. In other words, in a particular set of numbers, the standard of comparison or the units of measurement will undoubtedly be consistent. Seldom will the technician confront a situation in which he will have to deal with a combination of numbers that are not interrelated. This is an especially important consideration in carrying out the arithmetic operation of addition and subtraction of various fractions.

Before several *fractions* can be *added* or *subtracted*, they must have a *common denominator*. This principle can be demonstrated by using an architect's scale to lay out two consecutive dimensions, $\frac{3}{4}$ in. and $\frac{5}{16}$ in. (Fig. 2-4.)

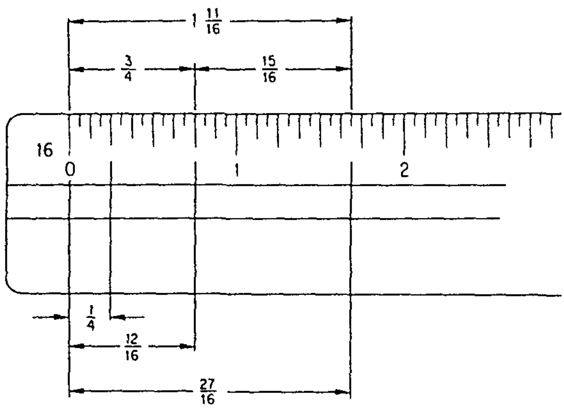


Figure 2-4

The procedure for laying off dimensions along a straight line and measuring their total length is called **graphical** or **geometric addition.**

$$\frac{3}{4}\text{ in.} + \frac{5}{16}\text{ in.} =$$

From the figure, it is evident that $\frac{3}{4}$ in. is equivalent to $\frac{12}{16}$ in. Thus,

$$\frac{12}{16}\text{ in.} + \frac{15}{16}\text{ in.} = \frac{27}{16}\text{ in., or } 1\frac{11}{16}\text{ in.}$$

This example serves as an introduction to the rule for addition and subtraction of fractions.

Fractions with a common denominator may be added or subtracted by combining the numerators according to the indicated operation and placing the sum or difference over the common denominator.

$$\frac{3}{7} + \frac{13}{7} - \frac{12}{7} = \frac{3 + 13 - 12}{7} = \frac{4}{7}$$

Fractions that do not have a common denominator first have to be converted to equivalent fractions with a common denominator They are then combined as previously stated

$$\frac{1}{2} + \frac{3}{4} - \frac{5}{8}, \quad \frac{1}{2} = \frac{4}{8}, \quad \frac{3}{4} = \frac{6}{8}$$

Thus,

$$\frac{1}{2} + \frac{3}{4} - \frac{5}{8} = \frac{4}{8} + \frac{6}{8} - \frac{5}{8} = \frac{5}{8}$$

A common denominator is not always determined by inspection The technician is aware that not all fractions appear as multiples of 2 as they have, more or less, in the discussion thus far The civil technician uses a drafting scale that is divided in multiples of 10 Most map and highway work is measured in feet and tenths of a foot The architect's plans are based on a scale of twelfths and multiples of twelve Mechanical draftsmen use a scale based on sixteenths Machinists and other craftsmen may use tools that read in terms of hundredths and thousandths The electronic technician and the chemical technician deal in millionths of certain units All of which suggests that not every problem can be solved by scaling or inspection What is presently needed is a simple method of finding the smallest quantity that is a multiple of all the fractions This will be called the *lowest common multiple*, **LCM**

The lowest common multiple of the denominators of several fractions becomes their lowest common denominators, **LCD**

EXAMPLE 2-B

Find the lowest common multiple of 4, 8, and 12

Solution

First, find prime factors of the given numbers

$$4 = (2)(2) = (2)^2$$
$$8 = (2)(2)(2) = (2)^3$$
$$12 = (3)(2)(2) = 3(2)^2$$

Note 2 appears twice as a factor of 4, twice as a factor of 12, and three times as a factor of 8, whereas 3 appears only once, as a factor of 12 The LCM of a set of numbers is the product of the several factors where each factor is included the maximum number of times it appears in any number in the set

Therefore, the LCM of 4, 8, and 12, is

$$(3)(2)^3 = 3 \times 8 = 24$$

The LCM, 24, is the smallest quantity that is divisible by every number in the set

There are many other common multiples of the given numbers 4, 8, and 12, such as:

$$(4)(12) = 48, \ (8)(12) = 96, \text{ and } (4)(8)(12) = 384$$

but none of these, obviously, is the lowest common multiple.

EXAMPLE 2-C:

Find the LCM of 18, 36, and 60.

Solution:

First, express given numbers in prime factors.

$$18 = (2)(3)(3) = (2)(3)^2$$
$$36 = (2)(2)(3)(3) = (2)^2(3)^2$$
$$60 = (2)(2)(3)(5) = (3)(5)(2)^2$$

Hence, $\quad \text{LCM} = (2)^2(3)^2(5) = (4)(9)(5) = 180$

Each factor must appear in the product as many times as it appears as a factor among the given numbers. *If the factor appears more often than the maximum, the result will still be considered a common multiple, but again, not the lowest common multiple.*

If the principle of the lowest common multiple is applied to the denominators of a set of fractions, a method thereby is established for finding the lowest common denominator. Initially, the objective was to come up with a procedure whereby fractions could be added or subtracted. This procedure has now been established.

EXAMPLE 2-D:

Combine the following fractions, as indicated.

$$\frac{5}{7} + \frac{8}{21} - \frac{1}{6} - \frac{3}{10} =$$

Solution:

First, find LCD.

$$7 = (7)(1) \text{ (prime number)}$$
$$21 = (3)(7)$$
$$6 = (3)(2)$$
$$10 = (5)(2)$$

Thus, $\quad \text{LCD} = (2)(3)(5)(7) = 210$

Next, convert given fractions to equivalent fractions having a denominator corresponding to LCD.

$$\frac{5}{7} = \frac{(5)(2)(3)(5)}{(7)(2)(3)(5)} = \frac{150}{210}$$

$$\frac{8}{21} = \frac{(8)(2)(5)}{(21)(2)(5)} = \frac{80}{210}$$

$$\frac{1}{6} = \frac{(1)(5)(7)}{(6)(5)(7)} = \frac{35}{210}$$

$$\frac{3}{10} = \frac{(3)(3)(7)}{(10)(3)(7)} = \frac{63}{210}$$

Therefore,

$$\frac{5}{7} + \frac{8}{21} - \frac{1}{6} - \frac{3}{10} = \frac{150}{210} + \frac{80}{210} - \frac{35}{210} - \frac{63}{210}$$

$$= \frac{150 + 80 - 35 - 63}{210}$$

$$= \frac{132}{210}$$

The answer should always be expressed in lowest terms

Hence,
$$\frac{132}{210} = \frac{(22)(6)}{(35)(6)} = \frac{22}{35}$$

EXAMPLE 2 E

Combine, as indicated

$$1\frac{3}{4} + 2\frac{5}{16} - \frac{7}{32} =$$

Solution

Convert mixed numbers to improper fractions

$$1\frac{3}{4} = 1 + \frac{3}{4} = \frac{4}{4} + \frac{3}{4} = \frac{7}{4}$$

$$2\frac{5}{16} = 2 + \frac{5}{16} = \frac{32}{16} + \frac{5}{16} = \frac{37}{16}$$

Thus,

$$1\frac{3}{4} + 2\frac{5}{16} - \frac{7}{32} = \frac{7}{4} + \frac{37}{16} - \frac{7}{32}$$

Next, find LCD and appropriate equivalent fractions

$$4 = (2)(2) = (2)^2$$

$$16 = (2)(2)(2)(2) = (2)^4$$

$$32 = (2)(2)(2)(2)(2) = (2)^5$$

Hence,

$$\text{LCD} = (2)^5 = 32$$

Therefore,

$$\frac{7}{4}=\frac{(7)(2)(2)(2)}{(4)(2)(2)(2)}=\frac{56}{32}$$

$$\frac{37}{16}=\frac{(37)(2)}{(16)(2)}=\frac{74}{32}\text{ and}$$

$$\frac{7}{4}+\frac{37}{16}-\frac{7}{32}=\frac{56}{32}+\frac{74}{32}-\frac{7}{32}=\frac{123}{32}$$

To convert $\frac{123}{32}$ to an improper fraction, divide accordingly:

$$\begin{array}{r} 3 \\ 32\;\overline{)\,123} \\ 96 \\ \hline 27 \end{array}\text{ Remainder}$$

Thus,

$$\frac{123}{32}=3\frac{27}{32}$$

Improper fractions are seemingly more convenient to use in most arithmetic operations. Furthermore, improper fractions are frequently used in technology to express certain principles. Gear ratios and velocity ratios are often given as $\frac{17}{5}$, $\frac{32}{21}$, ... In chemistry a solution may be prepared as 5 parts water to 3 parts acid. In civil technology, the slope of a road may be expressed as $\frac{12}{7}$. A draftsman, on the other hand, could not get by with dimensioning a drilled hole as $\frac{5}{4}$ in. The need will determine the form a number will assume.

EXERCISES 2-2

Find the least common multiple of the following sets of numbers (Ex. 1-10).

1. 2, 6, 9

2. 3, 5, 7, 11

3. 5, 7, 10

4. 7, 13, 17

5. 9, 12, 18, 36

6. 9, 16, 144

7. 11, 33, 77

8. 11, 154, 28

9. 5, 35, 70

10. 48, 60, 108

Complete the problems by finding the appropriate numerator or denominator of the respective equivalent fractions (Ex. 11-20).

11. $\frac{1}{2}=\frac{3}{\;}=\frac{\;}{18}$

12. $\frac{1}{\;}=\frac{\;}{13}=\frac{6}{78}$

13. $\frac{4}{\;}=\frac{\;}{12}=\frac{2}{3}$

14. $\frac{105}{\;}=\frac{98}{28}=\frac{63}{\;}$

15. $\frac{2}{11}=\frac{7}{\;}=\frac{\;}{121}$

16. $\frac{17}{\;}=\frac{\;}{17}=\frac{17}{\;}$

17. $\frac{4}{12} = \frac{\quad}{15} = \frac{\quad}{18}$

18. $\frac{108}{\quad} = \frac{\quad}{35} = \frac{132}{55}$

19. $\frac{5}{\quad} = \frac{6}{24} = \frac{7}{\quad}$

20. $\frac{a}{b} = \frac{2a}{\quad} = \frac{3a}{\quad}$

Perform the indicated operations and reduce the answers to lowest terms (Ex 21-30)

21. $\frac{2}{5} + \frac{3}{5} =$

22. $\frac{2}{3} + \frac{3}{4} + \frac{5}{6} =$

23. $\frac{5}{8} + \frac{3}{32} - \frac{7}{16} =$

24. $\frac{19}{64} + \frac{19}{16} - \frac{19}{32} =$

25. $5\frac{3}{8} - 4\frac{7}{16} =$

26. $3 + \frac{15}{32} - 2\frac{2}{3} =$

27. $\frac{4}{5} + \frac{11}{12} - \frac{6}{7} =$

28. $\frac{3}{8} + \frac{17}{40} + \frac{17}{32} - \frac{19}{20} =$

29. $1\frac{4}{21} + 3\frac{7}{15} - 4\frac{3}{35} =$

30. $2\frac{10}{11} + 3\frac{2}{3} - 4\frac{3}{4} =$

Find the value of the numerator or denominator that will satisfy the condition of the respective problem

31. $\frac{5}{9} + \frac{\quad}{63} = \frac{5}{3}$

32. $\frac{8}{7} - \frac{4}{\quad} = \frac{12}{35}$

33. $\frac{\quad}{5} - \frac{2}{3} = \frac{22}{165}$

34. $\frac{13}{10} + \frac{7}{5} + \frac{8}{\quad} = \frac{161}{30}$

Tolerance is defined as the allowable variation in dimensions of a manufactured item or part For dimensions given in fractions (mixed numbers), the tolerance is usually $\pm\frac{1}{64}$ in

Limits are referred to as the maximum or minimum permissible sizes within a given tolerance

EXAMPLE 2 F

Find the limits of the following dimension (Fig 2-5)

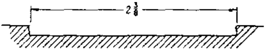


Figure 2-5

Solution

Since the tolerance is given in 64ths of an inch, the given dimension must be converted to a fraction with corresponding denominator, or

$$2\frac{3}{8} = 2\frac{24}{64}$$

Maximum limiting dimension

$$2\frac{24}{64} + \frac{1}{64} = 2\frac{25}{64}$$

Minimum limiting dimension:

$$2\frac{24}{64} - \frac{1}{64} = 2\frac{23}{64}$$

This means that the nominal dimension, $2\frac{3}{8}$ in., can vary between $2\frac{23}{64}$ in. and $2\frac{25}{64}$ in. and still meet engineering specifications.

Find the limits of the following dimensions (all dimensions in inches-tolerance $\pm\frac{1}{64}$).

35. $\frac{13}{64}$ *36.* $10\frac{3}{32}$ *37.* $5\frac{1}{16}$

38. $7\frac{1}{64}$ *39.* 10 *40.* $11\frac{63}{64}$

41. 12 *42.* $\frac{1}{64}$

2-5 MULTIPLICATION AND DIVISION OF FRACTIONS

The product of several fractions is another fraction whose numerator is the product of the respective numerators and whose denominator is the product of the respective denominators.

$$\frac{2}{3} \times \frac{5}{7} \times \frac{4}{11} = \frac{2 \times 5 \times 4}{3 \times 7 \times 11} = \frac{40}{231}$$

A whole number is treated as a fraction with a denominator equal to 1.

$$3 \times \frac{5}{7} = \frac{3}{1} \times \frac{5}{7} = \frac{15}{7}$$

In general form,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \ b \neq 0 \text{ and } d \neq 0$$

The quotient of two fractions is obtained by inverting the denominator fraction and multiplying it by the numerator fraction.

$$\frac{\frac{7}{2}}{\frac{5}{3}} = \frac{7}{2} \times \frac{3}{5} = \frac{21}{10}$$

EXAMPLE 2-G:

Find the product:

$$\frac{15}{8} \times \frac{56}{39} \times \frac{52}{105} =$$

Solution

The product can be obtained by multiplying numerators together and denominators together

$$\frac{15}{8} \times \frac{56}{39} \times \frac{52}{105} = \frac{(15)(56)(52)}{(8)(39)(105)} =$$

Before proceeding with the multiplication, each term of the numerator should be studied in terms of factors that may also appear in the denominator If this occurs, the procedure can be simplified by taking advantage of a fundamental principle $a/a = 1$, or like factors can be canceled

Thus,

$$\frac{(15)(56)(52)}{(8)(39)(105)} = \frac{(3 \times 5)(7 \times 8)(4 \times 13)}{(8)(3 \times 13)(3 \times 5 \times 7)} =$$

$$\frac{(\cancel{3})(\cancel{5})(\cancel{7})(\cancel{8})(4)(\cancel{13})}{(\cancel{8})(\cancel{3})(\cancel{13})(3)(\cancel{5})(\cancel{7})} = \frac{4}{3}$$

Therefore,

$$\frac{(15)(56)(52)}{(8)(39)(105)} = \frac{4}{3}$$

Alternate Solution

If recognized, like factors can be canceled without being reduced to prime factors

$$\frac{(\cancel{15})}{(\cancel{8})} \times \frac{\overset{7}{(\cancel{56})}}{\underset{3}{(\cancel{39})}} \times \frac{\overset{4}{(\cancel{52})}}{\underset{7}{(\cancel{105})}} = \frac{(4)(7)}{(3)(7)} = \frac{4}{3}$$

Note

8 is a factor of 56 (7×8)

$$\frac{15}{(\cancel{8})} \times \frac{\overset{7}{(\cancel{56})}}{39}$$

13 is a factor of both 39 and 52

$$39 = (3)(13) \text{ and } 52 = (4)(13)$$

$$\frac{56}{\underset{3}{(\cancel{39})}} \times \frac{\overset{4}{(\cancel{52})}}{105}$$

15 is a factor of 105 (7) (15)

$$\frac{(\cancel{15})}{8} \times \frac{56}{39} \times \frac{52}{\underset{7}{(\cancel{105})}}$$

(The 7 in the numerator and the 7 in the denominator could have been canceled in the first step.)

EXAMPLE 2-H:

Divide $5\frac{5}{9}$ by $\frac{35}{27}$

Solution:

First, convert $5\frac{5}{9}$ to an improper fraction.

$$5\frac{5}{9} = 5 + \frac{5}{9} = \frac{45}{9} + \frac{5}{9} = \frac{50}{9}$$

Next, apply the rule for division of fractions.

$$\frac{\frac{50}{9}}{\frac{35}{27}} = \frac{50}{9} \times \frac{27}{35} = \frac{\overset{10}{\cancel{50}}}{\cancel{9}} \times \frac{\overset{3}{\cancel{27}}}{\underset{7}{\cancel{35}}} = \frac{30}{7}$$

EXERCISES 2-3

Perform the indicated operations (Ex. 1-20).

1. $\frac{3}{10} \times \frac{5}{6} =$

2. $\frac{13}{9} \times \frac{27}{26} =$

3. $\frac{3}{10} \div \frac{5}{6} =$

4. $\frac{13}{9} \div \frac{26}{27} =$

5. $\frac{3}{2} \times \frac{7}{27} \times \frac{6}{49} =$

6. $1\frac{3}{5} \times 27 =$

7. $\frac{16}{15} \div \frac{48}{5} =$

8. $5\frac{15}{32} \div \frac{1}{8} =$

9. $10\frac{3}{5} \div 5\frac{3}{5} =$

10. $8\frac{1}{2} \times \frac{16}{17} \times 8\frac{1}{2} =$

11. $12\frac{3}{4} \div 9\frac{1}{16} =$

12. $16 \times 3\frac{5}{8} =$

13. $\frac{35}{32} \times \frac{27}{60} \times \frac{24}{63} =$

14. $\frac{90}{56} \times \frac{32}{65} \times \frac{91}{72} =$

15. $\frac{8}{9} \times \frac{35}{96} \times 108 =$

16. $144 \times \frac{1}{112} \times \frac{7}{9} =$

17. $\frac{119}{133} \div \frac{102}{171} =$

18. $\frac{115}{87} \div \frac{161}{174} =$

19. $\frac{105}{169} \times \frac{39}{40} \times \frac{104}{63} =$

20. $\frac{196}{625} \times \frac{25}{36} \times \frac{225}{49} =$

A fraction that contains one or more fractions in the numerator and one or more fractions in the denominator is referred to as a complex fraction

$$\frac{\frac{9}{7}-1}{\frac{11}{21}-\frac{1}{3}}$$

This mathematical expression shows up frequently in circuit-component relationships in the field of electronics

The total resistance of several resistors in a parallel circuit is given by the formula

$$\text{Total resistance, } R_t = \frac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}+\cdots},$$

where R_1, R_2, $\ldots$, refer to the resistors in the circuit

In the same discipline, the current, I, produced by several voltaic cells, n, grouped in parallel, is given by the formula

$$I=\frac{E}{r+\frac{R}{n}}$$

where E is the voltage, r, the total external resistance, and R, the total internal resistance of the cells

To simplify a complex fraction, reduce both numerator and denominator to a common fraction and then carry out the indicated division

EXAMPLE 2 1.

Simplify
$$\frac{\frac{9}{7}-1}{\frac{11}{21}-\frac{1}{3}}$$

Solution·

First, combine terms in the numerator and terms in the denominator, then divide

$$\frac{\frac{9}{7}-1}{\frac{11}{21}-\frac{1}{3}}=\frac{\frac{9}{7}-\frac{7}{7}}{\frac{11}{21}-\frac{7}{21}}=\frac{\frac{2}{7}}{\frac{4}{21}}=\frac{\cancel{2}}{\cancel{7}}\times\frac{\overset{3}{\cancel{21}}}{\underset{2}{\cancel{4}}}=\frac{3}{2}, \text{ or } 1\frac{1}{2}$$

Alternate Solution·

Another method of reducing the fractions is to multiply both numerator and denominator by the LCD of all the fractions This procedure is highly recommended for handling "complicated" complex fractions

The LCD for, 3, 7, and 21 is 21.

$$\frac{\frac{9}{7} - 1}{\frac{11}{21} - \frac{1}{3}} = \frac{\frac{9}{7} - 1}{\frac{11}{21} - \frac{1}{3}} \times \frac{21}{21} =$$

It is important to understand what occurs in the next step. The introduction of the fraction, $\frac{21}{21}$, means that **every term** in the **numerator** is **multiplied by LCD** and **every term** in the **denominator** is **multiplied by LCD.**

$$\frac{21\left(\frac{9}{7}\right) - 21(1)}{21\left(\frac{11}{21}\right) - 21\left(\frac{1}{3}\right)} = \frac{27 - 21}{11 - 7} = \frac{6}{4} = \frac{3}{2}$$

At all times throughout the solution, the main line of division must be firmly established and retained. The main line of division, the line that separates the given complex fraction into numerator and denominator, cannot be altered. All arithmetic operations involving the numerator are carried out above the line; likewise for the operation below the line.

EXAMPLE 2-J:

Simplify:

$$\frac{\frac{3}{8} + \frac{\frac{1}{2}}{3}}{5 - \frac{\frac{1}{4}}{\frac{3}{10}}}$$

Solution:

First, reduce the complex fraction in the numerator and the one in the denominator.

$$\frac{\frac{3}{8} + \frac{\frac{1}{2}}{3}}{5 - \frac{\frac{1}{4}}{\frac{3}{10}}} = \frac{\frac{3}{8} + \frac{3}{2}}{5 - \frac{10}{12}} = \frac{\frac{3}{8} + \frac{3}{2}}{5 - \frac{5}{6}}$$

Multiply numerator and denominator by the LCD, which is 24.

$$\frac{24\left(\frac{3}{8}\right) + 24\left(\frac{3}{2}\right)}{24(5) - 24\left(\frac{5}{6}\right)} = \frac{9 + 36}{120 - 20} = \frac{45}{100} = \frac{9}{20}$$

Simplify the following complex fractions (Ex 1-12)

1. $\dfrac{\frac{1}{2}}{1-\frac{1}{4}}$

2. $\dfrac{\frac{2}{3}}{\frac{3}{4}-\frac{1}{2}}$

3. $\dfrac{\frac{2}{3}}{\frac{3}{4}\times\frac{1}{2}}$

4. $\dfrac{4-\frac{2}{5}}{4-\frac{1}{10}}$

5. $\dfrac{\frac{13}{18}\times\frac{27}{39}}{\frac{17}{14}\times\frac{63}{51}}$

6. $\dfrac{3\frac{9}{16}-1\frac{1}{4}}{1\frac{7}{8}+\frac{1}{8}}$

7. $\dfrac{\frac{3}{5}+\dfrac{2}{\frac{1}{5}}}{\frac{9}{10}+\frac{4}{5}}$

8. $\dfrac{\frac{21}{34}\times\frac{85}{49}}{\frac{7}{5}-\frac{7}{15}}$

9. $\dfrac{\left(\frac{3}{5}\times\frac{10}{9}\right)-\frac{2}{15}}{\frac{7}{10}-\left(\frac{1}{2}\times\frac{2}{5}\right)}$

10. $\dfrac{3-1\frac{2}{3}}{4-\dfrac{\frac{1}{2}}{\frac{3}{4}}}$

11. $\dfrac{6\frac{3}{5}-6\frac{12}{20}}{\frac{6}{8}+\frac{3}{5}}$

12. $\dfrac{\frac{3}{8}\times\dfrac{\frac{1}{2}}{3}}{\frac{6}{5}\times\dfrac{\frac{1}{4}}{\frac{3}{10}}}$

13. Find R

$$R=\frac{1}{\frac{1}{250}+\frac{1}{300}+\frac{1}{75}}$$

14. Find R

$$R=\frac{1}{\frac{1}{27}+\frac{1}{45}+\frac{1}{30}}$$

15. Find I

$$I=\frac{1\frac{1}{2}}{150+\dfrac{\frac{1}{10}}{12}}$$

16. Find I.

$$I = \frac{9\frac{1}{3}}{220 + \frac{\frac{1}{5}}{9}}$$

REVIEW EXERCISES 2-5

1. Fill in the missing prime numbers.
   a. 11, 13, 17, . . . , 31.
   b. 43, 47, . . . , 59.
   c. 101, 103, 107, . . . , 151.

2. If two prime numbers are multiplied together will the product be a prime number? Explain briefly.

3. If one prime number is divided by a smaller prime number, will the quotient be an integer? Explain briefly.

Find the LCM of (Ex. 4-7):

4. 15, 45, 49

5. 42, 35, 22

6. 95, 85, 80

7. 54, 90, 81

8. Find dimension A (Fig. 2-6).

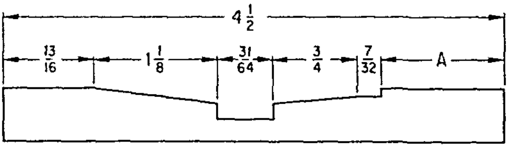


Figure 2-6

Perform the indicated operations (Ex. 9-24):

9. $\frac{3}{4} + \frac{9}{5} - \frac{3}{4} =$

10. $1\frac{2}{5} \times 2\frac{1}{3} \times 3\frac{3}{4} =$

11. $\frac{7}{8} + \frac{7}{16} + \frac{7}{32} =$

12. $\frac{9}{16} \times \frac{27}{64} \times \frac{256}{81} =$

13. $\frac{3}{5} + \frac{5}{5} + \frac{7}{5} - 3 =$

14. $\frac{65}{49} \times \frac{10}{9} \times \frac{42}{13} =$

15. $1\frac{13}{32} + 3\frac{5}{8} - 2\frac{1}{4} =$

16. $\frac{153}{114} \div \frac{102}{171} =$

17. $2\frac{7}{10} + 6\frac{3}{50} - 3\frac{9}{25} =$

18. $7\frac{2}{3} \times \frac{9}{46} =$

19. $9\frac{17}{50} - 4\frac{19}{50} + \frac{131}{125} =$

20. $\frac{6a}{\pi} + \frac{9a}{2\pi} =$

21. $\frac{3a}{25} + \frac{26a}{75} + \frac{80a}{150} - a =$

22. $\frac{3a}{4b} + \frac{5a}{6b} + \frac{8a}{9b} =$

23. $\frac{27b}{28} \times \frac{56}{54b} =$

24. $\frac{91}{a} \div \frac{117}{9a} =$

25. Find the limiting dimensions, tolerance $\pm\frac{1}{50}$ in

a $4\frac{7}{10}$ in

b 10 in

c $7\frac{49}{100}$ in

d $8\frac{49}{50}$ in

Simplify (Ex 26-28)

26. $$\frac{\frac{3}{7} \times \frac{91}{81} \times \frac{21}{104}}{\frac{5}{6} + \frac{7}{15}} =$$

27. $$\frac{\frac{5}{8} + \frac{2}{\frac{3}{4}}}{1 + \frac{1}{\frac{31}{32}} + \frac{\frac{3}{8}}{\frac{9}{16}}} =$$

28. $$\frac{1\frac{4}{21} + 3\frac{1}{9}}{3\frac{5}{27} - 2\frac{9}{35}} =$$

$$I = \frac{E}{r + \frac{R}{n}} \quad \text{(Ex 29-30)}$$

29. Find I if $E = 12$ volts (v), $r = 100$ ohms, $R = \frac{1}{20}$ ohms and $n = 8$

30. Find I if $E = 15$, $r = 15$, $R = \frac{1}{5}$, and $n = 3$

CHAPTER 3

Decimal Fractions and Percentage

A common fraction is defined as the indicated quotient of two integers (denominator other than zero). *If the denominator of a common fraction is* 10, *or some multiple of* 10, *the fraction is called a decimal fraction. If the denominator is* 100, *the numerator is expressed by a notation referred to as percentage, indicating the number of hundredths.*

Multiples of 10 can be expressed as powers of 10 with the use of exponents. This notation will be used in the development of certain principles associated with decimals.

$$100{,}000 = (10)(10)(10)(10)(10) = 10^5$$
$$10{,}000 = (10)(10)(10)(10) = 10^4$$
$$1{,}000 = (10)(10)(10) = 10^3$$
$$100 = (10)(10) = 10^2$$
$$10 = 10^1 \text{ (exponent of 1, seldom used)}$$

The decimal fraction is actually a convenient way of expressing the result of dividing the numerator by the denominator. The numerator becomes the quotient, whose digits are set off by a notation called a decimal point (.). There will be as many digits to the right of the decimal point as the power of the exponent in the denominator.

Fraction	Decimal Notation
$\frac{1{,}756}{10{,}000} = \frac{1{,}756}{10^4} =$	0.1756
$\frac{1{,}756}{1{,}000} = \frac{1{,}756}{10^3} =$	1.756
$\frac{1{,}756}{100} = \frac{1{,}756}{10^2} =$	17.56
$\frac{1{,}756}{10} = \frac{1{,}756}{10^1} =$	175.6

If the numerator has fewer digits than the power of 10 in the denominator, zeros are added to the left of the numbers in the quotient so that there will be as many digits (including zeros) as the power of 10

$$\frac{1{,}756}{100\,000} = \frac{1{,}756}{10^5} = \qquad 0\,01756$$

$$\frac{1{,}756}{1{,}000{,}000} = \frac{1{,}756}{10^6} = \qquad 0\,001756$$

The zero to the left of the decimal is used merely to emphasize the location of the decimal point

The terms **common fraction** and **decimal fraction** are used as a matter of distinction Generally, these mathematical concepts are referred to, simply, as **fractions** $\frac{1}{2}$, $\frac{7}{3}$, $\frac{15}{19}$, $\frac{49}{37}$, and **decimals**, 1 23, 0 57, 39 007,

The significance of the decimal point in relationship to the value of a digit (in a number) is illustrated in Fig 3 1

	10 000 Ten thousands	1,000 Thousands	100 Hund eds	10 Tens	1 0 Ones	Dec mal po nt	$\frac{1}{10}$ Ten hs	$\frac{1}{100}$ Hund edths	$\frac{1}{1000}$ Thousand hs	$\frac{1}{10000}$ Ten thousandths	
Number	0	0	0	0	0		0	0	0	0	
Twe ve					2						12
Twe ve and twe ve hund ed hs					2			2			12 12
Th ee hund ed n ne and two hund ed seven y f ve thousand hs			3	0	9		2	7	5		309 275
E gh and th y s x en housandths					8		0	0	3	6	8 0036
Four housand wenty and nine ten hs		4	0	2	0		9				4020 9
Forty e ght thousand th ee hundred seven y f ve and two thousand one hundred s x y n ne ten thousand hs	4	8	3	7	5		2	1	6	9	48375 2169

Figure 3 1

The numbers 12 12, 309 275, , are called **mixed decimals** They consist of a whole number (integer) and a decimal fraction Thus 12 12 is equivalent to $10 + 2 + \frac{1}{10} + \frac{2}{100}$

The illustration serves to demonstrate the need for proper alignment Before fractions can be added together they must be converted into equivalent fractions with a common denominator Likewise with decimals Tenths, $\frac{1}{10}$, are added to tenths, hundredths, $\frac{1}{100}$, are added to hundredths, and so on Therefore, proper placement assures addition of common denominations (denominators), (as in Ex 1-B)

Rule: *To subtract numerical quantities containing decimals, write the numbers involved in the operations so that the decimal points are directly under one another. Subtract respective digits, placing the decimal point in the difference or remainder (answer) to correspond with the other decimal points.* (Round off answer to meet criteria of significant figures.)

EXAMPLE 3-A:

Subtract 32.415 from 85.647.

Solution:

$$\begin{array}{r} 85.647 \\ \underline{32.415} \\ 53.232 \end{array}$$

EXAMPLE 3-B:

Subtract 2.56 from 12.875.

Solution:

$$\begin{array}{rl} 12.875 & \\ \underline{2.56} & \\ 10.315 & \quad \text{rounded off to } 10.32 \end{array}$$

EXERCISES 3-1

Add the following quantities, leaving the sum with corresponding significant figures.

1. $\begin{array}{r} 0.125 \\ \underline{0.65} \end{array}$ *2.* $\begin{array}{r} 1.25 \\ \underline{0.65} \end{array}$ *3.* $\begin{array}{r} 12.5 \\ \underline{65.0} \end{array}$

4. $\begin{array}{r} 156.3 \\ \underline{0.375} \end{array}$ *5.* $\begin{array}{r} 156.30 \\ \underline{0.375} \end{array}$ *6.* $\begin{array}{r} 156.300 \\ \underline{0.375} \end{array}$

7. $\begin{array}{l} 70.213 \\ \underline{13.0} \end{array}$ *8.* $\begin{array}{l} 0.0018 \\ \underline{0.132} \end{array}$ *9.* $\begin{array}{l} 0.0725 \\ \underline{0.058} \end{array}$

10. 13.762, 6.05, 9.125, 8.3

Subtract accordingly, leaving the difference or remainder with corresponding significant figures.

11. $\begin{array}{l} 0.725 \\ \underline{0.65} \end{array}$ *12.* $\begin{array}{l} 0.65 \\ \underline{0.125} \end{array}$ *13.* $\begin{array}{l} 6.5 \\ \underline{0.125} \end{array}$

14. $\begin{array}{r} 156.3 \\ \underline{0.375} \end{array}$ *15.* $\begin{array}{r} 156.3 \\ \underline{3.75} \end{array}$ *16.* $\begin{array}{l} 0.0027 \\ \underline{0.001} \end{array}$

17. $\begin{array}{l} 0.013 \\ \underline{0.0013} \end{array}$ *18.* $\begin{array}{r} 32.650 \\ \underline{32.065} \end{array}$ *19.* $\begin{array}{l} 1.0000 \\ \underline{1.00} \end{array}$

Regardless of all our precision tools and techniques, articles are still not manufactured to an exact dimension These items will vary in size within limiting dimensions established by a set of tolerances (Ex 3-C) These variations become an additional engineering concern when involvement extends to **clearance fits** of mating parts, where free movement is desired, such as a wheel rotating on a shaft, sliding motion, and engaging threads To insure against interference, the factor of **allowance** becomes critical *Allowance is defined as the predetermined difference between maximum material limits of mating parts* Clearance fit refers to the placement of limiting dimensions such that there is no possibility of interference when mating parts are assembled

EXAMPLE 3 C

The clearance fit for the mating parts in Fig 3-2 is calculated as follows

Allowance (tight fit) = Smallest Hole Size − Largest Shaft Size

allowance = 1 496 − 1 494 = 0 002 in

Loose fit = Largest Hole Size − Smallest Shaft Size

= 1 500 − 1 490 = 0 010 in

Tight fit 0 002 in clearance

Loose fit 0 010 in clearance

(The tight fit is called the allowance)

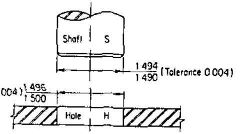


Figure 3 2

Using Fig 3-2, calculate the clearance fit, given the following dimensions, expressed in inches (indicate allowance)

	H	*S*
20.	1 750 / 1 752	1 745 / 1 743
21.	2 0050 / 2 0065	2 0030 / 2 0015

22. For an allowance of 0 005 in and tolerances of 0 005 in find the limiting dimensions of shaft and hole (Fig 3-2) if the maximum size of the shaft = 2 125 in Find maximum clearance also

23. For a loose fit of 0 0042, find the limiting dimensions for H and S (Fig 3-2) If the tolerances are 0 0015 in and the minimum size of the hole (H) is 1 0015 in , what is the allowance?

24. Find the clearance fit for the sliding mechanism in Fig 3-3 Indicate allowance

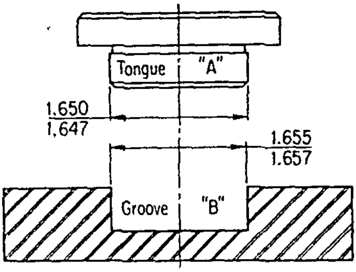


Figure 3-3

25. If the loose fit (Fig. 3-3) is designed to be 0.008 in. and the allowance is 0.002 in, what are the limiting dimensions of *A* and *B* if maximum size of slot is 2.125 in. Tolerance of tongue is 0.002 in.

26. If the allowance (Fig. 3-3) is 0.0015 in. and tolerances are held to 0.0005 in., find the limiting dimensions of *A* and *B* if maximum size of slot is 1.0550 in. Also find loose fit.

3-2 MULTIPLYING DECIMALS

Rule: *To multiply numerical quantities containing decimals, follow the procedure used in multiplying whole numbers. Point off in the product as many decimal places as there are in the combined sum of decimal places in the factors.*

EXAMPLE 3-D:

Multiply 2.162 by 7.41.

Solution:

$$\begin{array}{rl} 2.162 & \text{3 decimal places} \\ \underline{7.41} & \text{2 decimal places} \\ 2162 & \\ 8648 & \\ \underline{15134} & \\ 16.02042 & \overline{5}\text{ decimal places} \\ 16.02042 & \text{(rounded off)} = 16.02 \end{array}$$

EXAMPLE 3-E:

Multiply 0.012 by 0.009.

Solution:

$$\begin{array}{rl} 0.012 & \text{3 decimal places} \\ \underline{0.009} & \underline{3}\text{ decimal places} \\ 108 & \text{6 decimal places} \end{array}$$

If there are fewer digits in the product than are required for locating the decimal point properly, zeros are placed in front of the last digit on the left in the product until the decimal point is properly secured; thus:

```
    0 012
    0 009
 --------
 0 000108     6 decimal places
 0 000108     (rounded off) = 0 0001
```

Multiplying a number by 10 or any multiple of 10 will move the decimal point in the multiplicand the same number of places as there are zeros in the multiplier

EXAMPLE 3-F.

Multiply 175 26 by 100

Solution.

$$175\ 26 \times 100 = 175\ 26 = 17{,}526$$

EXAMPLE 3-G:

Multiply 175 26 by 1,000

Solution

$$175\ 26 \times 1000 = 175\ 260 = 175{,}260$$

```
    175 26      In order to move decimal point three places,
     10 00      another digit (zero) must be added to secure
 ---------      decimal point
    000 00
    0000 0
   00000
  17526
 ---------
 175260 00
```

Adding zeros beyond the last digit to the right of the decimal point will not change the numerical value of the number, only the precision of the number is affected

$$0\ 5 = 0\ 50 = 0\ 500 \qquad \text{(equivalent decimals)}$$

$$\frac{5}{10} = \frac{50}{100} = \frac{500}{1000} = \frac{1}{2} \qquad \text{(equivalent fractions)}$$

Again, 0 5 does not carry with it the same degree of precision of measurement as 0 500 (one decimal place as compared to three, tenths as compared to thousandths)

3-3 DIVIDING DECIMALS

Rule: *To divide numerical quantities containing decimals, follow the established procedure for carrying out the steps associated with long division of whole numbers (integers), placing the decimal point directly above the decimal point in the dividend*

EXAMPLE 3-H:

Divide 236.6 by 13.

Solution:

$$\begin{array}{r} 18.2 \\ 13\,\overline{)\,236.6} \\ \underline{13} \\ 106 \\ \underline{104} \\ 26 \\ \underline{26} \end{array} \qquad \text{(rounded off)} = 18$$

EXAMPLE 3-I:

Divide 0.2366 by 13.

Solution:

$$\begin{array}{r} 0.0182 \\ 13\,\overline{)\,0.2366} \\ \underline{13} \\ 106 \\ \underline{104} \\ 26 \end{array}$$

Since 13 is not contained in 2, a zero is placed above this digit to so indicate. Sometimes several zeros may have to be added in the quotient, before reaching sufficient digits, in dividend, that contain divisor.

EXAMPLE 3-J:

Divide 236.6 by 1.3.

Solution:

The divisor is not always an integer, so this condition must be accommodated. Thus, $236.6 \div 1.3$ can be written as $\dfrac{236.6}{1.3}$.

The value of a fraction remains unchanged when both numerator and denominator are multiplied by the same quantity. Thus,

$$\frac{236.6}{1.3} = \frac{236.6 \times 10}{1.3 \times 10} = \frac{2366}{13}, \text{ or}$$

$$1\,3_{\wedge}\,\overline{)\,236\,6_{\wedge}} = 13\,\overline{)\,2366} = 182.0$$

This demonstration based on the fundamental principles of fractions (Sec. 2-2), leads to the following rule: *The decimal point can be moved an equal number of places in the divisor and the dividend, respectively (in the same direction), without changing the value of the quotient.* The new position of the decimal point is then marked by a notation called a **caret** (∧).

EXAMPLE 3-K:

Divide 292.735 by 1.27.

Solution

Move decimal point two places to the right in both divisor and dividend and carry out the indicated operation

$$1\ 27\,\overline{)\,297\ 735} \qquad \text{becomes}$$

$$\begin{array}{r}
230\ 5 \\
127_{\wedge}\,\overline{)\,29273\ 5} \\
\underline{254}\ \ \ \ \ \ \\
387\ \ \ \ \\
\underline{381}\ \ \ \ \\
635\ \ \\
\underline{635}\ \
\end{array}$$

Note The decimal point is moved as many places as is needed to convert the divisor to a whole number

EXAMPLE 3 L

Divide 1 3 by 0 57

Solution

$$0\ 57\,\overline{)\,1\ 30} = \frac{1\ 3}{0\ 57} = \frac{(1\ 3)(100)}{(0\ 57)(100)} = \frac{130}{57}$$

$$\begin{array}{r}
2 \\
57_{\wedge}\,\overline{)\,130_{\wedge}} \\
\underline{114} \\
16
\end{array}$$

The answer (quotient) must be carried to one decimal place and practice suggests carrying one additional digit before rounding off Since the numerical value of a number is not affected by adding zeros to the right of the decimal point enough zeros must be incorporated to complete the problem

$$\begin{array}{r}
2\ 28 \\
57\,\overline{)\,130\ 00} \\
\underline{114}\ \ \ \ \ \\
160\ \ \ \\
\underline{114}\ \ \ \\
460\ \\
\underline{456}\ \\
4\
\end{array} \qquad \text{(rounded off)} = 2\ 3$$

EXERCISES 3 2

Perform the indicated operations and round off your answer to correspond to three significant figures (Ex 1 20)

1. $3.125 \times 6 =$

2. $31.25 \times 0.6 =$

3. $21.25 \times 0.06 =$

4. $3.1416 \times 700 =$

5. $32.5 \times 100 =$

6. $\frac{22}{7} \times 700 =$

7. $0.00625 \times 2.5 \times 0.625 =$

8. $1.500 \times 1.5 \times 15.0 =$

9. $0.0616 \times 1.0 \times 7.35 =$

10. $1.000 \times 1 \times 0.100 \times 0.010 =$

11. $0.3125 \div 2 =$

12. $3.125 \div 0.2 =$

13. $31.25 \div 0.002 =$

14. $33.625 \div 0.100 =$

15. $43.9824 \div 3.1416 =$

16. $48.9824 \div \frac{22}{7} =$

17. $3.875 \div 1.25 =$

18. $3\frac{7}{8} \div 1\frac{1}{4} =$

19. $367.9654 \div 100 =$

20. $367.9674 \div 0.00100 =$

21. One pound (lb) is equivalent to 453.6 g. What is the weight of 1 g in equivalent units of 1 lb.

22. How many pounds are there in a bar of silver weighing 2,266.0 g?

23. Find the displacement of a four-cylinder diesel engine with a bore, d, of 4.875 in. and a stroke, h, of 5.50 in.

$$\text{Displacement} = 4\left(\frac{\pi d^2}{4}\right)h$$

24. Moving the decimal point one place to the right is equivalent to multiplying by 10. Explain what happens if the decimal point is moved to the left one or more places.

Two very familiar relationships, fundamental to electric circuit analysis, are Ohm's law and the equation for finding power. Ohm's law states that the voltage across any component of a circuit is proportional to the product of the current through that component and the resistance of that component, or $E = IR$, where E is the voltage, I the current (amps) and R the resistance (ohms). Power, P, in watts is defined by the equation $P = EI$.

25. Find the voltage across a resistance of 16.4 ohms when the current through the resistance measures 6.7 amps.

$$E = IR;\ E = 6.7 \times 16.4 = 109.88, \text{ rounded off to } 109.9 \text{ volts.}$$

26. What voltage is required for a current of 4.2 amps to flow through a resistance of 26.4 ohms?

27. The resistance of a vacuum tube is 12.15 ohms when it draws 0.906 amps. Find the voltage.

28. A measurement on a meter shows 220 v across a component when the current is 15.8 amps. What is the resistance of the component?

29. Find the current through a resistance of 63.5 ohms when the voltage across the resistance is 7.50 volts.

30. How much power is expended by a current of 0 375 amps produced by a voltage of 110 volts?

31. What is the power rating of a lamp that draws 1 83 amps with a voltage of 110 volts

32. The power rating of an electrical appliance is 220 watts (w) This unit draws 3 2 amps What is the voltage?

33. Compare the amount of current each of these household lamps will draw with a voltage of 120 volts

a 40 watts
b 100 watts
c 150 watts
d 240 watts

3-4 CHANGING FRACTIONS TO DECIMALS

Frequently, fractions may prove more convenient to apply than decimals

Very often the technician will be involved with a problem containing data given in several units of measurement Along with this, numbers associated with the units may also appear in various forms ($\sqrt{2}$, π, 3 1416, $\frac{22}{7}$, 1 732,) Numbers can be viewed as a language of convenience Some times it will be more favorable to work with the radical form of $\sqrt{2}$ than its decimal equivalent, 1 414 Circumstances surrounding the nature of the problems and local engineering practices will prevail

In any event, the technician should be able to use fractions and decimals interchangeably

Rule: *To change a fraction to a decimal, divide the numerator of the fraction by the denominator, retaining in the quotient the number of figures deemed significant*

EXAMPLE 3 M

The application of this rule may lead to an equivalent fraction that is either exact or approximate

If the equivalent decimal **terminates**, the results are **exact**

a Express $\frac{3}{4}$ as an equivalent decimal

$$\begin{array}{r} 0\,75 \\ 4\,\overline{)\,3\,00} \\ \underline{2\,8} \\ 20 \\ \underline{20} \end{array}$$

here, the decimal **terminates**

Thus, $\frac{3}{4} = 0\,75$, or $\frac{3}{4} = 0\,750$, as it usually appears in terms of linear measurements

If the equivalent decimal **does not terminate**, the result is only an **approximation.**

b. Express $\frac{5}{13}$ as an equivalent decimal.

$$\begin{array}{r}
0.384615 \\
13\,\overline{)\,5.000000} \\
\underline{39} \\
110 \\
\underline{104} \\
60 \\
\underline{52} \\
80 \\
\underline{78} \\
20 \\
\underline{13} \\
70 \\
65
\end{array}$$

It should be evident that this decimal will never terminate, no matter how far the operation is carried.

Thus, since $\frac{5}{13}$ is not equal (exactly) to 0.384615, the decimal is rounded off to some predetermined standard (three places in this example) and indicated as:

$\frac{5}{13} \approx 0.385$, where the symbol $\approx$, means *approximately equal to.*

Since the **decimal does not terminate**, it cannot be expressed as the quotient of two integers and is therefore considered an **irrational number.**

c. Convert $\frac{5}{3}$ to an equivalent decimal.

$$\begin{array}{r}
1.6666 \\
3\,\overline{)\,5.0000} \\
\underline{3} \\
20 \\
\underline{18} \\
20 \\
\underline{18} \\
20 \\
\underline{18} \\
20 \\
\underline{18} \\
2
\end{array}$$

This is another example of a quotient that will never terminate. Notice that the same digit (after the first) keeps repeating. This condition is referred to as a **repeating decimal,** and sometimes is written as 1.6666 . . . .

Thus, $\frac{5}{3} \approx 1.667$ (three places).

A **repeating decimal** can be considered as an example of a **rational number.**

If the prime factors of the denominator of a common fraction, reduced

to lowest terms, contain multiples of 2, 5 or 10 only the decimal (equivalent) will eventually terminate. Notice that these conditions are associated with the principal units of measurement that will be used by the technician

$$\frac{3}{16}\text{ in} = 0\ 1875\text{ in}$$

$$\frac{31}{64}\text{ in} = 0\ 484375\text{ in}$$

$$\frac{13}{50}\text{ in} = 0\ 26\text{ in}$$

$$\frac{9}{10}\text{ ft} = 0\ 90\text{ ft}$$

To express a decimal (fraction) as an equivalent (common) fraction, simply apply the definition of decimal fraction

EXAMPLE 3-N

Express 0 625 as a common fraction in lowest terms

Solution

By definition

$$0\ 625 = \frac{625}{1{,}000} = \frac{\cancel{(5)}\cancel{(5)}\cancel{(5)}(5)}{\underset{2}{\cancel{(10)}}\,\underset{2}{\cancel{(10)}}\,\underset{2}{\cancel{(10)}}} = \frac{5}{8}$$

EXERCISES 3-3

Express the following (common) fractions or mixed numbers as equivalent decimals carried to three places (thousandths)

1. $\frac{7}{8}$	*2.* $\frac{9}{7}$	*3.* $\frac{7}{9}$
4. $1\frac{15}{16}$	*5.* $\frac{31}{32}$	*6.* $\frac{63}{64}$
7. $\frac{63}{100}$	*8.* $3\frac{7}{10}$	*9.* $\frac{36}{144}$
10. $\frac{21}{20}$	*11.* $7\frac{5}{16}$	*12.* $\frac{210}{90}$
13. $\frac{91}{119}$	*14.* $\frac{51}{75}$	*15.* $\frac{171}{247}$

Express the following decimals as equivalent fractions in lowest terms

16. 0 35	*17.* 0 670	*18.* 2 125
19. 0 4	*20.* 0 111	*21.* 2 45
22. 0 245	*23.* 0 130	*24.* 0 006

25. 9.60	*26.* 0.96	*27.* 0.960
28. 1.000	*29.* 0.101	*30.* 0.999

3-5 PERCENTAGE

When the denominator of a common fraction is 100, the numerator can be represented by a symbol, %, called **percentage** or just **per cent.** *Per cent actually means by (per) the hundred or hundredths* (cent). Thus 20% can be defined by the fraction $\frac{20}{100}$, or the decimal 0.20.

This mathematical concept is used in all areas of technology. Resistors and capacitors are labeled by color code expressing rating tolerances in per cent. Solutions are given in terms of per cent concentration. Factors of safety in stress design can be interpreted as percentages of maximum loading. Per cent or percentage can be used to compare how a part or portion relates to the whole (per cent = part/whole).

However meaningful the notation 20% may appear, it cannot be involved computationally while in this particular form. To be operational, per cent must be expressed as an equivalent fraction or decimal.

Rule: *To change or convert per cent to a fraction, drop the % symbol* (sign) *and divide the number, expressed as per cent, by* 100 *or multiply the number by* $\frac{1}{100}$.

EXAMPLE 3-O:

Change 20% to an equivalent fraction or decimal.

Solution:

$$20\% = \frac{20}{100} = \frac{1}{5}$$

$$20\% = \frac{20}{1} \times \frac{1}{100} = \frac{1}{5}$$

Rule: *To change or convert per cent to a decimal, drop the % sign and multiply the number, expressed as per cent, by* 0.01 *or divide the number by* 100. (*This in essence is the same as moving the decimal point of the number two places to the left.*)

EXAMPLE 3-P:

Convert 20% to an equivalent decimal.

Solution:

$$20\% = 20 \times 0.01 = 0.20$$

$$20\% = \frac{20}{100} = 100\,\overline{)\,20.00}\quad \begin{array}{r} 0.20 \\ \hline 20.00 \\ \underline{20.0} \\ 00 \end{array}$$

Rule: *To convert or change a fraction or decimal to an equivalent percentage, multiply by* 100 *and affix the % symbol to the product.*

EXAMPLE 3-Q·

Change $\frac{3}{4}$ and 0 75 to per cent

Solution.

$$\frac{3}{4} \times 100 = \frac{300}{4} = 75\%$$

$$0\ 75 \times 100 = 75\%$$

The conversion of per cent to a fraction or decimal provides a useful and orderly arrangement that can be used in other arithmetic computations

EXAMPLE 3-R

Find 8% of 25

Solution

Express the given per cent as an equivalent decimal or fraction

$$8\% = \frac{8}{100} = 0\ 08$$

Next, multiply the given number by the equivalent fraction or decimal

$$\frac{8}{100} \text{ of } 25 = \frac{8}{100} \times 25 = 2$$

or

$$8\% \text{ of } 25 = 0\ 08 \times 25 = 2\ 00 \text{ (rounded off)} = 2$$

On occasion, per cent is used to compare one item in terms of another item, such as the chemical composition of the various substances that make up a compound or sample In this context, per cent is regarded as a ratio of part/whole

Ratios are expressed as fractions, thus the rule for converting fractions to per cent apply

$$\text{per cent} = \frac{\text{part}}{\text{whole}} \times 100$$

EXAMPLE 3-S

A mixture weighing 32 0 g contains 4 0 g of sulphur What per cent of the sample is made up of sulphur?

Solution

$$\%\text{ sulphur} = \frac{4\ 0}{32\ 0} \times 100 = \frac{1}{8} \times 100 = 12\ 5\%$$

EXERCISES 3-4

Change the following decimals or fractions to per cent (%)

1. $\frac{1}{10}$ *2.* 0 10 *3.* $\frac{2}{5}$

4. $\frac{4}{3}$
5. 0.40
6. 0.67
7. $\frac{7}{8}$
8. 0.05
9. 0.005
10. $\frac{1}{100}$
11. $\frac{1}{1000}$
12. 0.0075
13. $1\frac{1}{4}$
14. $2\frac{1}{2}$
15. 1.000
16. 2.50
17. 2.05
18. 2.15
19. $2\frac{15}{100}$
20. 12.5
21. $13\frac{1}{100}$

Convert each of the percentages to fractions and decimals.

22. 10%
23. 50%
24. 75%
25. 40%
26. 15%
27. 17%
28. 99%
29. 100%
30. $12\frac{1}{2}\%$
31. $62\frac{1}{2}\%$
32. 120%
33. 150%
34. 200%
35. 1%
36. 2%
37. 2.5%
38. 3.25%
39. 0.10%
40. 0.01%
41. $\frac{1}{2}\%$
42. $20\frac{4}{5}\%$
43. 20.8%
44. $44\frac{3}{5}\%$
45. $\pi\%$

Find the indicated percentages of the respective quantities.

46. 20% of 100
47. 15% of 50
48. 100% of 1
49. 33% of 33
50. $12\frac{1}{2}\%$ of 64
51. $33\frac{1}{3}\%$ of 90
52. 200% of 100
53. 150% of 100
54. 11% of $\frac{9}{11}$
55. 42% of 30.2
56. 1.5% of 0.015
57. $4\frac{3}{4}\%$ of $7\frac{1}{8}$
58. 40% of $\frac{8}{5}$
59. $62\frac{1}{2}\%$ of $\frac{8}{5}$
60. 90% of $\frac{100}{9}$
61. $67\frac{2}{3}\%$ of $\frac{27}{12}$
62. $87\frac{1}{2}\%$ of 19.2
63. $6\frac{2}{3}\%$ of 4.20
64. $9\frac{1}{11}\%$ of 99
65. 99% of 1.0
66. 1% of 99
67. 100% of 0.0
68. The alcohol content of a certain solution is 44% by volume. How many ounces of alcohol are contained in a 16.0-ounce sample?
69. Find the allowable limits of an 3,500 ohm resistor with a tolerance of $\pm 5\%$.

70. A solution contains 15 g of sodium hydroxide per 1,000 cm^3. How many grams of sodium hydroxide are contained in 100 cm^3 of this solution?

71. Concrete can be mixed to contain $17\frac{1}{2}\%$ cement by weight. Find the (cement) composition by weight in a 3,750-lb load.

72. The per cent elongation of a supporting member is 0.01% for 1°F rise in temperature. How many inches will this 12-ft support (original length) stretch with a temperature rise of 40°F?

73. Structural nickel steel may contain 3.5% nickel and 0.45% carbon. Find the weight of each of these materials in a ton of this steel.

74. A nickel-chromium steel has a composition of 2.75% nickel, 0.80% chromium, and 0.65% manganese. What is the weight of each of these alloys found in $3\frac{1}{2}$ tons of this steel.

75. The error in measurement of a height gauge was found to be 0.010 in. in 12.000 in. What is the per cent error of this instrument?

76. *What per cent of 100 is 33?*

77. What per cent of 33 is 100?

78. $12\frac{1}{2}\%$ is what per cent of 25?

79. 17 is what per cent of 50?

80. Find the percentage of water in an alcohol solution of 50 cm^3, if the amount of alcohol is found to be 42 cm^3.

81. If the error in measurement is 0.20 in. for every 10.00 in., find the per cent error (recall, per cent = part/whole × 100).

82. Find the per cent error in a stop watch if it is off 0.02 seconds (sec) in every minute.

83. What per cent of the circumference of a circle is its diameter?

84. A ton of steel was found to contain 7.2 lb of carbon. Find the per cent of carbon.

85. It has been found that a ton of gangue produced 6 ounces of gold. What percentage of the gangue was gold? How many tons of the earthy material would be required to yield $2\frac{1}{2}$ lb of gold?

86. A resistor was rated as 680 ohms. When checked out in the lab it measured 660 ohms. What is the per cent error of the rating?

REVIEW EXERCISES 3-5

Perform the indicated arithmetic operations.

1. $27.06 + 13.701 - 12.505 =$

2. $18.928 - 10.025 + 3.750 - 2.125 =$

3. $1.001 + 10.01 - 1.001 - 10.01 =$

4. $\dfrac{32.16 \times 7.01 \times 0.021}{8.4 \times 70.1 \times 0.0021} =$

5. $\dfrac{(17.76 \div 1.6)}{0.02} =$

6. $\dfrac{\frac{7}{12}}{3 + \frac{\frac{1}{2}}{\frac{1}{3}}} =$

7. $\dfrac{3}{\frac{2}{\frac{1}{2}} - 1} =$

8. $\dfrac{\frac{1}{2}}{2 - \frac{\frac{1}{2}}{2}} =$

9. $\dfrac{\frac{7}{5} - 1}{1 + \frac{1}{\frac{1}{2}} + \frac{1}{\frac{\frac{1}{2}}{2}}}$

10. $\dfrac{8.24 \times 30.21 \times 0.001}{30.00 \times 0.040} =$

11. $\dfrac{1.000 \times 0.100 \times 0.010}{0.001 \times 10.000} =$

12. $1.25 \times 3.016 \times 4.315 =$

13. $\dfrac{6.250 \times 5.00}{31.25} =$

14. $\dfrac{22}{7} \div \pi =$

15. One of the most useful equations employed by the structural technician is a simple relationship, which can be used to solve complex problems.

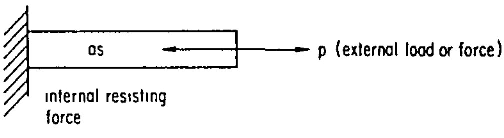


Figure 3-4

$P = as$; Load = area × stress; stress is defined as the internal resisting force. Area is the cross-sectional area of the member subjected to the load P (in pounds), and is given in square inches; therefore, $P = as$ is defined as:

External Load = Total Internal Resisting Force

EXAMPLE 3-T:

Find the tensile load that produces a stress of 8,000 pounds per square inch (lb/in.^{2}) in a steel rod if the area is $2\frac{1}{2}$ in.^{2}?

Solution:

$$P = 2\frac{1}{2} \times 8{,}000 = \frac{5}{2} \times 8{,}000 = 20{,}000 \text{ lb}$$

16. A compression stress of 100 lb/in ² is developed in a 2 in × 4 in timber Compute the magnitude of the load

17. A hollow cast iron post is used to support a load of 2,500 lb Find the stress developed (Fig 3-5)

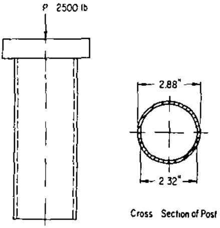


Figure 3-5

18. If the stress developed in the supporting member of Fig 3-5 is 4,000 lb/in ², what is the corresponding load?

19. What area is needed (Fig 3-5) to support a load of 7,500 pounds with a design stress held to 4,500 lb/in ²?

20. Convert to fraction or decimal

a 0 390	b 1 250	c $3\frac{7}{8}$
d 1 01	e $12\frac{12}{12}$	f 1 000

21. Complete the table

	Fraction	Decimal	Per cent
a	$\frac{5}{6}$	0 833	$83\frac{1}{3}\%$
b	______	0 375	______
c	______	______	92%
d	$\frac{3}{64}$	______	______
e	______	1 20	______
f	______	______	0 5%
g	$1\frac{7}{16}$	______	______
h	______	______	2 00%
i	$\frac{9}{20}$	______	______

22. Find $83\frac{1}{3}\%$ of 3.6.

23. Find $6\frac{5}{8}\%$ of 320.

24. Find 200% of 1.0.

25. Find 20% of 50%.

26. What per cent of 60 is 10?

27. What per cent of 42 is 42?

Density is defined as the ratio of mass/volume or density = mass/volume ($d = m/v$) expressed as grams per cubic centimeter or pounds per cubic foot.

EXAMPLE 3-U:

What is the density of a lead sample that weighs 28.50 g with a volume of 2.50 cm^3?

Solution:

$$d = \frac{m}{v} = \frac{28.50 \text{ g}}{2.50 \text{ cm}^3} = 11.4 \frac{\text{g}}{\text{cm}^3}$$

28. Find the density of a metallic specimen that weighs 62.4 lb and measures 1.1 cubic feet (ft^3).

29. The density of water is 62.4 pounds per cubic foot (lb/ft^3). Find the volume of 100 lb of water.

30. A wood block measures 5.08 cm × 7.62 cm × 25.4 cm and weighs 515 g. What is its density?

31. Gold has a density of about 19.3 grams per cubic centimeter (g/cm^3). Find the weight of a cylindrical gold rod that is 10.2 cm in length with a diameter of 2.5 cm.

32. Silver has a density of 10.5 g/cm^3. Find the weight of a cylindrical silver rod with the same dimensions as the rod in problem 31.

33. By what per cent is the gold rod heavier than the silver rod (problems 31-32)?

34. 1,600 lb of sea water was found to contain approximately 780 ounces of salt, 40 ounces of magnesium, 30 ounces of sulphur, 18 ounces of potassium, 12 ounces of calcium, 2.0 ounces of bromine, and 16 ounces of other minerals. Find the per cent composition of the various minerals found in this sample of sea water.

35. What percent of the sea water is made up of minerals?

36. The diameter of a circle is what per cent of its circumference?

37. One inch is equal to 2.54 centimeters (1 in. = 2.54 cm). What part of an inch is 1 cm?

38. How many centimeters are there in $\frac{7}{8}$ of an inch?

39. A reading of 452.4 cm was found to have an error of 0.03 cm. Find the per cent of error.

40. An 800 g sample of lunar rock was found to have the following composition, Oxygen, 472 g, Silicon, 144 g, Aluminum, 72 g, and other elements, 112 g. Find the percentage composition, by weight, of each element.

CHAPTER 4

Scientific Notation, Exponents, and the Slide Rule

On occasion, to complete a mathematical or scientific process, the need arises to incorporate very large numbers or extremely small numbers associated with physical units. For example, the velocity of light, $v = 299{,}790{,}000$ meters per second (m/sec), or 186,285 miles per hour (miles/hr), or; the mass of a particle, $m = 0.000000125$ g. Although electrical calculators and electronic computers make short work of just about any quantity, it is still awkward, almost meaningless, to carry excessive digits at any point.

4-1 SCIENTIFIC NOTATION

An extreme quantity can be expressed in a simplified form, referred to as **scientific notation**. This is accomplished by expressing the given number, N, as the product $N = M \times 10^m$, where M is a number made up of significant digits representing the given number. Furthermore, M is larger than 1 but less than 10. The letter m is called an *exponent* and indicates the number of times the same quantity (base) appears as a *multiple* or *factor*.

$$10 \times 10 \times 10 \times 10 = 10^4, \text{ here } m = 4$$

$$7 \times 7 \times 7 \times 7 \times 7 = 7^5, \text{ exponent is } 5$$

Thus,

$$299{,}790{,}000. = 29{,}979 \times 10 \times 10 \times 10 \times 10 = 29{,}979 \times 10^4$$

or

$$2997.9 \times 10^5 = 299.79 \times 10^6 = 29.979 \times 10^7 = 2.9979 \times 10^8$$

The last expression is referred to as **scientific notation**; the others are called **standard notation.** Usually, no more than 3 significant digits are used to represent a number in scientific notation.

Hence,

$$299{,}790{,}000 \text{ meters per second} = 3\ 00 \times 10^{8} \text{ m/sec}$$

Notice that the exponent represents (numerically) the number of places that the original decimal point was moved to the left

Numbers that are less than 1 are represented in scientific notation by a negative exponent The negative exponent indicates the number of places the decimal point was moved to the right

Thus,

$$0\ 000000125 \text{ g} = 1\ 25 \times 10^{\ 7} \text{ g}$$

EXAMPLE 4 A

Express, 3,726,100 000 in scientific notation, rounded to 3 significant figures

Solution

$$3{,}726{,}100{,}000. = 3\ 73 \times 10^{9}$$

Notice that the decimal point was moved 9 places to the left In terms of the general form

$$N = M \times 10^{m},\ M = 3\ 73 \text{ and } m = 9$$

EXAMPLE 4 B

Express, 0 0000356 in scientific notation with 2 significant digits

Solution

To express 0 0000356 in scientific notation, the decimal point must be moved 5 places to the right This is reflected by the exponential notation -5

Thus,

$0\ 0000356 = 3\ 56 \times 10^{\ 5}$, rounded to $3\ 6 \times 10^{\ 5}$ to meet stated conditions

EXERCISES 4 1

Express the following quantities in scientific notation with 3 significant figures

1. 32,000	*2.* 0 00032	*3.* 717,000
4. 0 000717	*5.* 50,520 000	*6* 826,000 000
7. 0 0000445	*8.* 0 0000446	*9.* 0 0000444
10. 955,000	*11.* 105,000 000	*12* 364 600 000
13. 0 0000355	*14.* 2,005,000 000	*15* 2,060,000

16. 0.050

17. 0.500

18. 5.0

19. 0.00010

20. 1

21. 1.00×10^1

22. 0.03×10^2

23. 30.3×10^{-2}

24. 0.00526×10^{-1}

25. 5.26×10^{-1}

26. 50.26×10^1

27. $56.34 \times 10^2 \times 10^{-2}$

28. $6400 \times 10^3 \times 10^{-2}$

29. $10^3 \times 15.29 \times 10^{-4}$

30. 100.001

4-2 LAWS OF EXPONENTS

To be able to change or rewrite a number in exponential form is no finished accomplishment. Exponents do, however, perform an important function in the fundamental mathematical process, especially in multiplying and dividing quantities bearing extreme measurements.

Looking at the number 10,000 and using the principle of factoring, 10,000 can be expressed in several different forms:

$10{,}000 = 10 \times 10 \times 10 \times 10 = 100 \times 100 = 10 \times 1{,}000 = 1.0 \times 10^4$

Furthermore, $100 = 10 \times 10 = 10^2$ and $1{,}000 \times 10 \times 10 \times 10 = 10^3$

It follows that $10{,}000 = 1.0 \times 10^2 \times 10^2 = 1.0 \times 10^1 \times 10^3 = 1.0 \times 10^4$. Apparently, $10^2 \times 10^2 = 10^4$ and $10^1 \times 10^3 = 10^4$.

This observation leads to the **Law of Exponents for Multiplication,** which states: *To find the product of several factors having a common base, add the exponents of the factors to obtain the exponent of the product.*

In general form,

$$a^m \times a^n = a^{m+n} \text{ where } a \neq 0$$

EXAMPLE 4-C:

Find the product of $3^2 \times 3^4 \times 3^5$.

Solution:

$$3^2 \times 3^4 \times 3^5 = 3^{2+4+5} = 3^{11}$$

In this example, 3 is considered the base with exponents 2, 4, and 5, respectively.

In science and mathematics, 3^{11} is considered acceptable form. Numerically, however,

$$3^{11} = 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 177{,}147$$

EXAMPLE 4-D:

Multiply $2^2 \times 3^3$.

Solution:

An indicated product, such as $2^2 \times 3^3$, cannot be combined according to the Law of Exponents since 2 and 3 do not form a common base. To

complete the problem, it is necessary to expand each factor and then multiply the products

$$2^2 = 2 \times 2 = 4 \text{ and } 3^3 = 3 \times 3 \times 3 = 27$$

Thus,

$$2^2 \times 3^3 = 4 \times 27 = 108$$

Exponents appear quite frequently in mathematical expressions involving division as well as multiplication.

For example, the quotient of 10,000 ÷ 100 can be computed by using exponential notation.

$$\frac{10{,}000}{100} = \frac{1.0 \times 10^4}{1.0 \times 10^2} = \frac{1.0 \times 10^2 \times 10^2}{1.0 \times 10^2} = 1.0 \times 10^2, \text{ or } 100$$

Notice that $10^2/10^2 = 1$. *Any number, other than zero, divided by itself is equal to 1.*

Apparently,

$$\frac{10{,}000}{100} = 1.0 \times 10^{4-2} = 1.0 \times 10^2 = 1.0 \times 10^2$$

This illustration can best be summarized by the **Law of Exponents for Division**. *To find the quotient of two factors having a common base, subtract the exponent in the divisor from the exponent in the dividend to obtain the exponent of the quotient.* In general form,

$$\frac{a^m}{a^n} = a^{m-n}, \text{ if } m \text{ is larger than } n \ (m > n)$$

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \text{ if } n > m, \ a \neq 0$$

By definition, $a^m = 1/a^{-m}$ or $a^{-n} = 1/a^n$, and $a^0 = 1$, again $a \neq 0$.

By way of illustration, $a^{-n} = 1/a^n$, $10^{-2} = 1/10^2$. Recall that in scientific notation, 10^{-2} could be used to represent a number such as $0.02 = 2.0 \times 10^{-2}$. But 0.02 can also be expressed as

$$0.02 = \frac{2.0}{100} = \frac{2.0}{10^2}$$

Thus,

$$2.0 \times 10^{-2} = \frac{2.0}{10^2}, \text{ or } 10^{-2} = \frac{1}{10^2}$$

Similarly ($a^0 = 1$),

$$\frac{a^m}{a^n} = a^{m-n}, \text{ hence } \frac{5^3}{5^3} = 5^{3-3} = 5^0$$

But

$$\frac{5^3}{5^3} = \frac{5 \times 5 \times 5}{5 \times 5 \times 5} = \frac{125}{125} = 1$$

Thus,

$$5^0 = 1, \text{ or } a^0 = 1; \text{ also, } 10^0 = 1, -13^0 = 1, \ldots .$$

EXERCISES 4-2

Perform the indicated operations and simplify results.

1. $10^2 \times 10^3$

2. $5^4 \times 5^2$

3. $\dfrac{7^3 \times 7^0}{7^3}$

4. $8^5 \div 8^3$

5. $10^3 \times 10^2$

6. $7^3 \times 3^3$

7. $2^3 \times 3^3$

8. $14^3 \div 14^2$

9. $10{,}000 \div 200$

10. $\dfrac{8^2}{8 \times 8^0}$

11. $\dfrac{10^3}{5^3}$

12. $\dfrac{15^4}{5^4}$

13. $\dfrac{16^5}{8^4}$

14. $\dfrac{21^2}{7^2}$

15. $\dfrac{10^2 \times 10^{-2}}{5}$

16. $\dfrac{5}{10^2 \times 10^{-2}}$

17. $\dfrac{5^{3-2}}{5^{3+2}}$

18. $\dfrac{10^3 + 10^2}{10^5}$

19. $\dfrac{10^{10}}{10^{18}} + 10^2$

20. $10^2 + 10^2$

21. $5^3 + 2^2$

22. $\dfrac{7^3 - 7^2}{7^2 - 7}$

23. $\dfrac{0.001 \times 0.0001}{100}$

24. $\dfrac{1{,}000 \times 10{,}000 \times 0.0000001}{0.001 \times 1{,}000}$

25. $\dfrac{1.0 \times 10.00 \times 100.00}{1 \times 0.1 \times 0.01}$

4-3 MULTIPLYING AND DIVIDING WITH EXPONENTS

Rule: *To multiply quantities expressed in exponential form, multiply the numerical portion of the multiplicand and the multiplier, following the usual procedure for multiplication. To complete the problem, combine the exponents according to the laws of exponents.*

EXAMPLE 4-E:

Find the product of $67{,}000 \times 1{,}300$

Solution:

Express each number in scientific or exponential form

$$67{,}000 = 6.7 \times 10^4 \text{ and } 1{,}300 = 1.3 \times 10^3$$

Thus,

$$67{,}000 \times 1{,}300 = (6.7 \times 10^4) \times (1.3 \times 10^3) = (6.7 \times 1.3) \times (10^4 \times 10^3)$$
$$= 8.71 \times 10^{4+3} = 8.71 \times 10^7, \text{ rounded to } 8.7 \times 10^7$$

Rule: *To divide quantities expressed in exponential form, divide the numerical portion of the dividend by the numerical portion of the divisor, following the established arithmetic procedure for division. Proceed, then, to combine the exponents according to the laws of exponents.*

EXAMPLE 4-F

Simplify

$$\frac{376{,}000}{0.000188}$$

Solution:

Express the dividend and divisor in scientific notation

$$376{,}000 = 3.76 \times 10^5 \text{ and } 0.000188 = 1.88 \times 10^{-4}$$

Thus,

$$\frac{376{,}000}{0.000188} = \frac{3.76 \times 10^5}{1.88 \times 10^{-4}} = \frac{3.76}{1.88} \times \frac{10^5}{10^{-4}}$$

Furthermore,

$$\frac{3.76}{1.88} = 2$$

and

$$\frac{10^5}{10^{-4}} = 10^5 \times 10^4, \text{ since } \frac{1}{10^{-4}} = 10^4$$

Therefore,

$$\frac{376{,}000}{0.000188} = 2.00 \times 10^{5+4} = 2.00 \times 10^9$$

By long division,

$$0.000188_\wedge \overline{)\,376000\,000000_\wedge} \quad \text{quotient } 2000\,000000 \text{ or } 2.00 \times 10^9$$

$$\begin{array}{r} 376 \\ \hline 0 \end{array}$$

EXAMPLE 4-G:

Simplify

$$\frac{360{,}000 \times 0.008}{280}.$$

Solution:

First, express each number in scientific notation.

$360{,}000 = 3.60 \times 10^5$, $0.008 = 8 \times 10^{-3}$, and $280 = 2.80 \times 10^2$

Thus,

$$\frac{360{,}000 \times 0.008}{280} = \frac{(3.60 \times 10^5) \times (8 \times 10^{-3})}{2.80 \times 10^2} = \frac{3.60 \times 8}{2.80} \times \frac{10^5 \times 10^{-3}}{10^2}$$

Furthermore,

$$\frac{3.60 \times 8}{2.80} = 10.3; \frac{10^5 \times 10^{-3}}{10^2} = \frac{10^5}{10^2 \times 10^3} = \frac{10^5}{10^5} = 1$$

Hence,

$$\frac{360{,}000 \times 0.008}{280} = 10.3, \text{ rounded to } 10$$

Exponential notation was introduced to facilitate mathematical computation. Its usage should not be limited strictly to scientific notation, however, but should be extended to include standard notation as well. Conditions may suggest that a quantity such as 424,000 be written as 424×10^3 rather than as 4.24×10^5. Again, usage will determine the proper form. Furthermore, it is advisable to work with positive exponents, taking advantage of the identity

$$a^{-1} = \frac{1}{a} \quad \text{or} \quad \frac{1}{a^{-1}} = a$$

EXAMPLE 4-H:

Simplify

$$\frac{1{,}690 \times 0.0119}{91 \times 0.00017}.$$

Solution:

First, where appropriate, standard notation will be used.
$1{,}690 = 169 \times 10$, $0.0119 = 119 \times 10^{-4}$, and $0.00017 = 17 \times 10^{-5}$

Thus,

$$\frac{1{,}690 \times 0.0119}{91 \times 0.00017} = \frac{(169 \times 10) \times (119 \times 10^{-4})}{91 \times (17 \times 10^{-5})}$$

$$= \frac{\overset{13}{\cancel{169}} \times \overset{7}{\cancel{119}}}{\underset{7}{\cancel{91}} \times \cancel{17}} \times \frac{10 \times 10^{-4}}{10^{-5}} = 13 \times \frac{10 \times 10^{5}}{10^{4}} = \frac{13 \times 10^{6}}{10^{4}}$$

$$= 13 \times 10^{2} = 1{,}300$$

Note $169 = 13 \times 13$, $119 = 17 \times 7$, $91 = 13 \times 7$

EXERCISES 4-3

Perform the indicated operations and simplify

1. $\dfrac{35{,}000}{0.70}$ *2.* $0.035 \times 1{,}200 \times 0.0015$

3. $425{,}000 \times 0.00035 \times 120$ *4.* $1{,}500{,}000 \times 15{,}000 \times 0.00015$

5. $(36 \times 10^{8}) \times (24 \times 10^{6})$ *6.* $1{,}130 \times 0.250 \times 0.00416$

7. $(42.5 \times 10^{6}) \times (42.5 \times 10^{-6})$ *8.* $(110 \times 10^{-6}) \times (2.60 \times 10^{1})$

9. $\dfrac{27{,}000 \times 0.0008 \times 110}{0.0064 \times 5.5 \times 3{,}000}$ *10.* $\dfrac{0.0032 \times 0.1015}{0.0050 \times 0.040}$

11. $\dfrac{45 \times 10^{5} \times 92 \times 10^{2} \times 121}{11 \times 10^{-2} \times 23 \times 10^{2} \times 90 \times 10^{3}}$

12. $\dfrac{138{,}000 \times 142{,}000 \times 51{,}000}{71{,}000 \times 230{,}000}$

13. $\dfrac{12.5 \times 0.0035 \times 0.064 \times 22{,}000}{0.0011 \times 25.0 \times 0.0070 \times 800}$

14. $\dfrac{1}{3.025 \times 0.0021 \times 1.01}$ *15.* $\dfrac{1}{2 \times 800 \times 0.000005}$

It is important to remember that the laws of exponents, thus far, pertain to the **multiplication and division** of exponential forms, **not to addition and subtraction**
Hence,

$$6^{3} \times 6^{2} = 6^{5} = 7{,}776$$

whereas,

$$6^{3} + 6^{2} = (6 \times 6 \times 6) + (6 \times 6) = 216 + 36 = 252$$

A word of caution concerning reducing expressions such as

$$\frac{6^{-4} + 6^{-2}}{6^{-3}}$$

$$\frac{6^{-4} + 6^{-2}}{6^{-3}} \neq \frac{6^{3}}{6^{4} + 6^{2}}$$

According to the identity, $a^{-1} = 1/a$, $6^{-4} = 1/6^4$, $6^{-3} = 1/6^3$, and $6^{-2} = 1/6^2$
Thus,

$$\frac{6^{-4} + 6^{-2}}{6^{-3}} = \frac{\frac{1}{6^4} + \frac{1}{6^2}}{\frac{1}{6^3}} = \frac{\frac{1}{6^4} + \frac{6^2}{6^4}}{\frac{1}{6^3}} = \frac{\frac{1 + 6^2}{6^4}}{\frac{1}{6^3}} = \frac{6^3(1 + 6^2)}{6^4} = \frac{1 + 6^2}{6}$$

Note:

$$\frac{1 + 6^2}{6} \neq \frac{1 + \cancel{6} \times 6}{\cancel{6}} \neq 1 + 6 = 7$$

However,

$$\frac{1 + 6^2}{6} = \frac{1 + 36}{6} = \frac{37}{6}$$

4-4 THE SLIDE RULE

The **slide rule**, Fig. 4-1, is constructed and developed according to the principles of **logarithms. Logarithms**, Chapter 14, are defined in terms of **exponents.** Furthermore, slide-rule computations, by and large, are considered reliable to 3 digits and most frequently involve quantities expressed in scientific notation. There are various operations that can be carried out on the slide rule, but the discussions here will include only those that are used most often by the technician: multiplication, division, squaring, and extracting square roots.

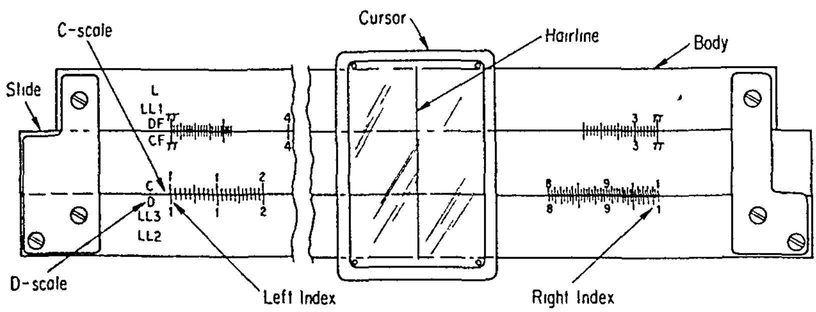


Figure 4-1

All the scales on the slide rule indicate digits but do not set the decimal point, and it is this feature that requires greatest attention. The *C* and *D* scales, Fig. 4-2, are used in multiplication and division and are identical. Several readings are illustrated.

The first reading, 103, can be interpreted as 1.03×10^m, or 0.103, 10.3, 103, 1030, 0.00103, ...

To find the product of 2 numbers:

1. Move the cursor until the hairline locates one of the numbers on the *D* scale;

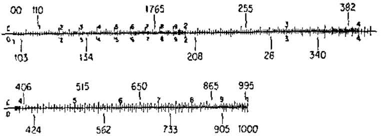


Figure 4-2

2 Move the slide until either the left index or right index coincides with the hairline (and first number),

3 Move the cursor until the hairline locates the second number on the C scale,

4 Product appears on the D scale, beneath hairline (below the second number on the C scale)

Several products will be indicated in which one of the numbers contains the digits 15 (Fig 4-3)

To divide two numbers:

1 Move the cursor until the hairline locates the dividend on the D scale,

2 Move the slide, C scale, until the divisor coincides with the hairline (which now places divisor over dividend),

3 Quotient appears under the index on the D scale

Several illustrations appear in Fig 4 4 in which the quotient is represented by the digit 2

EXAMPLE 4 1

Using the slide rule, find the product of 32 2 × 565

Solution

Express each number in scientific notation

$$32\ 2 = 3\ 22 \times 10 \text{ and } 565 = 5\ 65 \times 10^2$$

Thus,

$$32\ 2 \times 565 = (3\ 22 \times 10) \times (5\ 65 \times 10^2) = (3\ 22 \times 5\ 65) \times (10^3)$$

On the slide rule (Fig 4-5), 3 22 × 5 65 appears as 322 × 565 = 182

By inspection, 3 22 × 5 65 can be approximated as a product slightly larger than 15

Therefore,

$$(3\ 22 \times 5\ 65) \times 10^3 = 18\ 2 \times 10^3 = 1\ 82 \times 10^4$$

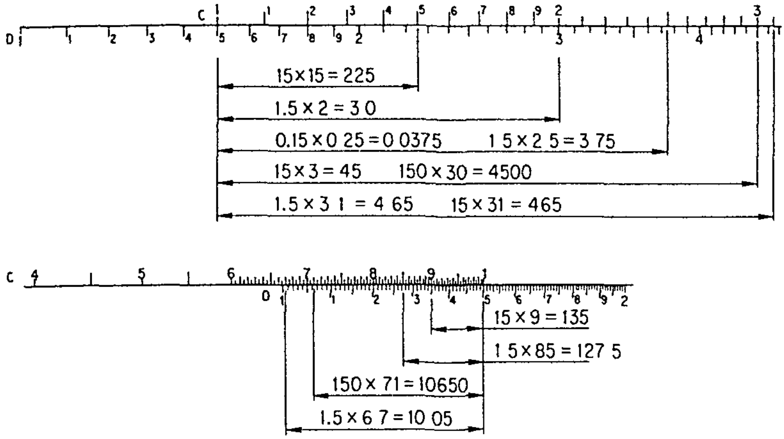


Figure 4-3

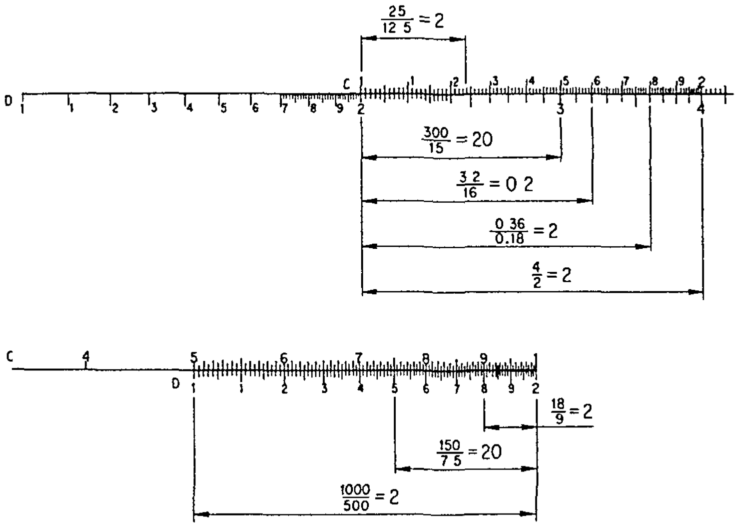


Figure 4-4

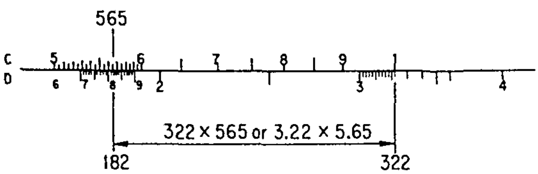


Figure 4-5

EXAMPLE 4-J:

Find the quotient of 66.4/0.426.

Solution:

$$66.4 = 6.64 \times 10 \text{ and } 0.426 = 4.26 \times 10^{-1}$$

Thus,

$$\frac{66.4}{0.426} = \frac{6.64 \times 10}{4.26 \times 10^{-1}} = \frac{6.64}{4.26} \times 10^{2}$$

On the slide rule (Fig 4 6), 6 64/4 26 appears as 664/426 = 156

By inspection, 6 64/4 26 can be estimated as 1 5

Therefore,

$$\frac{66\ 4}{0\ 426} = \frac{6\ 64}{4\ 26} \times 10^2 = 1\ 56 \times 10^2$$

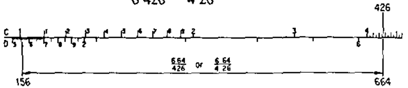


Figure 4-6

Approximations locate the decimal point, recall that the slide rule relates only in terms of digits

EXERCISES 4-4

Using the slide rule, perform the indicated arithmetic computation Leave the answer in scientific notation

1. 36×42

2. $3\ 6 \times 4\ 2$

3. $18\ 2 \times 8\ 42$

4. $6\ 05 \times 0\ 195$

5. $4{,}062 \times 308$

6. $17\ 9 \times 1\ 01$

7. $0\ 0246 \times 0\ 905$

8. $1\ 57 \times 10^3 \times 2\ 22 \times 10^5$

9. $369 \times 3\ 69$

10. $4\ 68 \times 10^4 \times 4\ 68 \times 10^{-4}$

11. $\frac{42}{14}$

12. $\frac{14}{42}$

13. $\frac{1{,}096}{966}$

14. $\frac{34\ 2}{2\ 38}$

15. $\frac{0\ 855}{0\ 713}$

16 $\frac{1}{99\ 5}$

17. $\frac{3\ 1416}{25\ 4}$

18. $\frac{88\ 2}{0\ 036}$

19. $\frac{79\ 4}{0\ 794}$

20. $\frac{1}{84{,}820}$

Many computations in engineering involve both multiplication and division On the slide rule, such computations are usually carried out by alternating the operations, starting with division followed by multiplication and continuing in that order until the problem is completed

EXAMPLE 4 K

Simplify

$$\frac{3\ 6 \times 48\ 2 \times 688}{6\ 80 \times 50\ 3 \times 99\ 3}$$

Solution:

$$\frac{3.6 \times 48.2 \times 688}{6.80 \times 50.3 \times 99.3} = \frac{(3.6 \times 4.82 \times 6.88) \times (10^3)}{(6.80 \times 5.03 \times 9.93) \times (10^2)}$$

Start by dividing $\frac{3.6}{6.80}$, which leads to 53 on the *D* scale. Multiply this quotient by 4.82 and divide the product by 5.03, which leads to 508 on the *D* scale. Multiply 508 by 6.88 and then complete the computation by dividing the last product by 9.93. The result appears on the *D* scale as 352. The decimal point is located as previously.

The entire computation can be summarized accordingly:

$$\left(\frac{3.6}{6.80}\right) \times \frac{(4.82 \times 6.88) \times (10^3)}{(5.03 \times 9.93) \times (10^2)} = 0.53 \times \frac{(4.82 \times 6.88) \times 10}{(5.03 \times 9.93)}$$

Further,

$$\left(\frac{0.53 \times 4.82}{5.03}\right) \times \left(\frac{6.88}{9.93}\right) \times 10 = 0.508 \times \left(\frac{6.88}{9.93}\right) \times 10$$

Finally,

$$\left(\frac{0.508 \times 6.88}{9.93}\right) \times 10 = 3.52$$

This procedure minimizes the number of moves required to complete a computation of this nature. The chance of error is thus reduced.

EXERCISES 4-5

Using the slide rule, perform the indicated computation. Leave the answer in scientific notation.

1. $3.6 \times 4.8 \times 5.2$

2. $\frac{3.6 \times 4.8}{5.2}$

3. $\frac{1}{9.5 \times 8.46}$

4. $\frac{13.7 \times 38.4}{27.4 \times 19.2}$

5. $\frac{0.048 \times 3.06}{14.4 \times 7.12}$

6. $\frac{46.2 \times 38.4 \times 96.8}{13.3 \times 53.9}$

7. $\frac{100.2 \times 68.68}{2.01 \times 40.08 \times 8.055}$

8. $\frac{9.0 \times 3.14 \times 31.46 \times 15.12}{16.8 \times 7.56 \times 3.14}$

9. $\frac{33{,}000 \times 550 \times 1{,}414}{32.2 \times 980}$

10. $\frac{3.14 \times (6.24)^2}{9.88 \times (3.14)^2}$

The *A* and *D* scales are arranged such that the *A* scale defines the square of a number directly below it on the *D* scale, which also means that the *D* scale will define square roots of numbers appearing on the *A* scale (Fig. 4-7).

The square of a number, as indicated, can be determined by going

directly from the *D* scale to the *A* scale However, the inverse operation that of finding the square root of a number, is not quite as simple or direct Some care must be exercised in this procedure since the *A* scale is made up of 2 complete scales whereas the *D* scale is a single scale

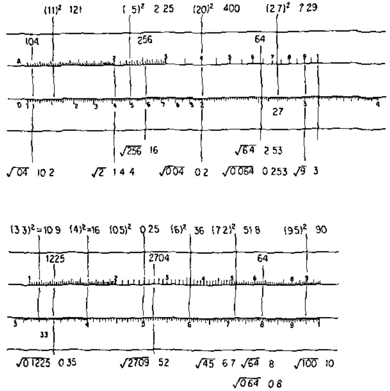


Figure 4 7

The square root of a number can be determined by following these guidelines

1 If the number whose square root is to be determined is larger than 1 and
   a if there are an odd number of digits to the left of the decimal point locate the given number on the left half of the *A* scale and read directly below on the *D* scale for the digits representing the answer,
   b if there are an even number of digits to the left of the decimal point locate the number on the right half of the *A* scale and proceed as before
2 If the number is less than 1, count the number of places to the right of the decimal point to locate the first significant digit Then,
   a for an odd number of places, locate the given number on the right half of the *A* scale,
   b for an even number of places, locate the given number on the left half of the *A* scale and proceed as previously indicated

Thus,

LEFT HALF	(A SCALE)	RIGHT HALF
$\sqrt{6.4} = 2.53$,		$\sqrt{64.} = 8$
1 digit (odd)		2 digits (even)
$\sqrt{0.064} = 0.253$,		$\sqrt{0.64} = 0.8$
2 places (even)		1 place (odd)

EXERCISES 4-6

Using the slide rule, compute the value of the following expressions.

1. $\sqrt{9.8}$

2. $\sqrt{98.0}$

3. $\sqrt{0.98}$

4. $\sqrt{0.098}$

5. $\sqrt{102}$

6. $\sqrt{642}$

7. $\sqrt{4800}$

8. $\sqrt{52{,}900}$

9. $\sqrt{0.169}$

10. $\sqrt{0.00169}$

11. $\sqrt{3.2 \times 7.8}$

12. $\dfrac{\sqrt{512 \times 48}}{\sqrt{12 \times 14}}$

13. $(3.6)^2$

14. $(7.02)^2$

15. $(0.55)^2$

16. $\dfrac{(81 \times 36)^2}{(15 \times 9.4)^2}$

17. $\dfrac{(7.24)^2(\sqrt{81.2})}{(9.01)^2(\sqrt{52.4})}$

18. $\dfrac{\sqrt{54} \times (54)^2}{7.35 \times 29.0}$

REVIEW EXERCISES 4-7

1. Approximately $2.45 \times 10^{-10}\%$ of sea water is gold. How many tons of sea water would be needed to produce 1 ounce (oz) of gold?

2. One horsepower (hp) is defined as doing work at the rate of 33×10^3 foot-pounds per minute (ft-lb/min). How much work is done in 30 sec by a 0.75-hp motor?

3. How much work is done by a 54.2-hp auto engine during a 2.25-hour (hr) drive?

4. An Angstrom (Å) is a unit of length equivalent to 10^{-8} cm, or 1Å = 10^{-8} cm. Express a wave length of 0.0000007 cm in terms of Angstroms.

5. Express a wave length of 2.25 Å in centimeters.

EXAMPLE 4-L:

The coefficient of linear expansion of a solid is defined as the increase in unit length when the temperature is increased 1 degree (decrease in length for

decrease in temperature) For a certain grade of steel, the coefficient of linear expansion is 24×10^{-6} °F Unit length refers to any convenient standard, such as foot, inch, etc

Find the total expansion of a 30-ft steel tie rod as the temperature changes from a morning low of 67°F to an afternoon high of 92°F Calculate the length of the rod when the temperature reaches 92°F

Solution

The linear expansion for a rise of 1°F is 24×10^{-6} per unit of length If the unit of length is 1 in, a rise of 1°F would mean that the steel tie rod would elongate (expand or stretch) an amount equivalent to 24×10^{-6} in, or 0 000024 in If the unit happened to be in terms of centimeters, the elongation would then be 24×10^{-6} cm for every degree rise in temperature (°F)

For this illustration, the unit of length will be 1 ft Total rise in temperature = 92°F − 67°F = 25°F

$$\text{Unit expansion (elongation per foot)} = \text{temperature rise} \times \text{coefficient}$$
$$= 25 \times 24 \times 10^{-6}\ \text{ft} = 600 \times 10^{-6}\ \text{ft}$$
$$= 60 \times 10^{-5}\ \text{ft}$$

$$\text{Total expansion} = \text{original length} \times \text{unit expansion}$$
$$= 30 \times (60 \times 10^{-5}) = 1{,}800 \times 10^{-5}\ \text{ft}$$
$$= 18 \times 10^{-3} = 0\ 018\ \text{ft}$$

$$\text{Length of rod at 92°F} = \text{original length} + \text{total expansion}$$
$$= 30 + 0\ 018 = 30\ 018\ \text{ft}$$

6. Calculate the total expansion of the steel tie rod in Ex 4-L in inches

7. The coefficient of linear expansion for iron is 20×10^{-6} increase in units of length per 1°F Find the linear expansion of a 36-in base made of iron as the temperature changes 32°F

8. The coefficient of volume expansion of water is $2\ 16 \times 10^{-6}$ increase in unit volume per Fahrenheit degree Calculate the total expansion of 7×10^{3} gallons (gal) of water as the temperature rises from 65°F to 90°F

9. For the same water tank and capacity of problem 8, find the change in volume of water as the temperature drops from 72°F to 40°F

10. The speed of light is usually taken as $18\ 6 \times 10^{4}$ miles per second (miles/sec) At this rate, how long does it take the rays of the sun to reach the earth (distance earth to sun = 93×10^{6} miles)?

Strain is defined as a **deformation** (elongation) due to an axial load The unit strain is equal to the total elongation (or shortening) of a member divided by the original length of the member

$$\text{unit strain} = \frac{\text{total elongation}}{\text{original length}}$$

EXAMPLE 4-M:

Find the unit strain of the tie rod in Ex. 4-L at a temperature of 92°F.

Solution:

original length = 30 ft
total elongation = 0.018 ft

$$\text{unit strain} = \frac{0.018}{30} = \frac{18 \times 10^{-3}}{3 \times 10} = \frac{18}{3} \times 10^{-4} = 6 \times 10^{-4} = 0.0006$$

Strain, like the coefficient of linear expansion, carries no units, such as feet, meters, inches, etc. It is a ratio of similar units and remains the same regardless of the standard of measure. The only restriction is that, once established, the units remain consistent throughout the problem.

The results of Ex. 4-M remain the same even if the unit of measurement, selected originally, was in inches (12 in. = 1 ft).

original length = 30 × 12 = 360 in.
total elongation = 0.018 × 12 = 0.216 in. (rounded to) = 0.22 in.

$$\text{unit strain} = \frac{22 \times 10^{-2}}{36 \times 10} = \frac{11}{18} \times 10^{-3} = 0.61 \times 10^{-3}$$
$$\text{(rounded to)} = 0.60 \times 10^{-3} = 0.0006$$

11. Find the unit strain for the cast iron base in problem 7.

12. An aluminum wire, 48 in. long, stretches a total of 12×10^{-3} in. when subjected to a pull of 60 lb. Find the unit strain. What is the developed stress if the area of the wire is 24×10^{-2} in.^{2}?

13. A concrete support is subjected to a load that produces a unit strain of 8×10^{-4}. By how much is this support shortened if the length, before loading, was 18 ft?

14. A **capacitor** (sometimes referred to as a condensor) is made up of several plates or strips separated by an insulating material. The **resistance** of a capacitor to the **flow** of **alternating current** is called **capacitive reactance**. The formula for calculating the capacitive reactance, denoted by X_c is given as:

$$X_c = \frac{1}{2\pi f C}\,\text{(ohms)}$$

f is the frequency of the current given in cycles per second (cps). C is the capacitance of the capacitor, given in a unit called a farad (f).

EXAMPLE 4-N:

What is the capacitive reactance of a capacitor rated at 2.5×10^{-10} f when the frequency is 12.5×10^5 cps?

Solution:

$$X_c = \frac{1}{2\pi \times 2.5 \times 10^{-10} \times 12.5 \times 10^5}$$

$$= \frac{1 \times 10^{10}}{2\pi \times 2.5 \times 12.5 \times 10^{5}} = \frac{1 \times 10^{5}}{2\pi \times 2.5 \times 12.5}$$

$$= \frac{1 \times 10^{5}}{196.25} = 510 \text{ ohms}$$

15. What is the capacitive reactance of a 32×10^{-10} f capacitor at a frequency of 1.6×10^{6} cps?

16. Find the capacitive reactance of a capacitor rated at 4.1×10^{-11} f at a frequency of $\frac{7}{22} \times 10^{7}$ cps $(\pi = \frac{22}{7})$

17. The capacitive reactance of a capacitor rated at 8.2×10^{-3} f was found to be 2.1 ohms. At what frequency did this occur?

18. There are 6.02×10^{23} molecules in 56 g of iron. How many molecules are there in 28 g of iron?

19. There are 3.01×10^{23} molecules in 9 g of water. How many molecules are there in 18 g of water?

20. Which would contain more molecules, 14 g of iron or 4.5 g of water?

21. Simplify

$$\frac{\dfrac{1}{2 \times 10^{4}}}{\dfrac{1}{20} + \dfrac{1}{2 \times 10^{2}} + \dfrac{1}{2 \times 10^{3}}}$$

22. Simplify

$$\frac{\dfrac{20}{\frac{1}{10}} + \dfrac{1}{\frac{1}{10^{2}}} - \dfrac{3}{\frac{1}{10^{2}}}}{\dfrac{30}{\frac{1}{10^{2}}} + \dfrac{1}{\frac{1}{10^{3}}} - \dfrac{3.999}{\frac{1}{10^{3}}}}$$

23. Evaluate

$$\frac{(\sqrt{3.14}) \times (9.2)^{2} \times (34.2)}{(3.14)^{2} \times \sqrt{9.2} \times \sqrt{34.2}}$$

24. Evaluate

$$\frac{32.7 \times 79.6 \times \sqrt{0.36}}{(2.06)^{2} \times 6.004 \times \sqrt{0.036}}$$

25.

$$f = \frac{1}{2\pi\sqrt{LC}}$$

Find f (cps) if $L = 0.0003$ henry and $C = 0.0000075$ f

CHAPTER 5

Dimensional Analysis

Numbers are a convenient way of representing units of measure relating, basically, to the three fundamental physical quantities: **length, mass,** and **time.** All of these properties are involved with the laws of nature and other scientific premises. By and large, every conceivable technological symbol or concept is a related component of length, mass, and time.

5-1 UNITS OF MEASUREMENT

The technician will be directly involved with some form of measurement and designed application of the characteristics of weight, area, volume, velocity, energy, electricity, heat, temperature, force, pressure, and many other physical phenomena. All of these are developed in units of measurement that relate back to the fundamental quantities.

Along with quantities associated with a standard of measurement, a number representing a dimension, there are *dimensionless numbers.* A most familiar example of a dimensionless number is the ratio of the circumference of a circle to its diameter (π). Of interest here is that a **dimensionless number carries no unit of measure** since it is a ratio of two similar quantities: feet divided by feet, meters over meters, inches/inches, and so on. Other examples of dimensionless numbers are those associated with specific gravity, unit strain, per cent, and the trigonometric functions. Symbols defining these properties are not dependent on the units of measurement other than the fact that the units have to be identical. These expressions are important factors contributing to meaningful relationships among several physical quantities. *Working with units of measurement in terms of physical relationships is called dimensional analysis.* The derivation of many complex technological formulas can be attributed to dimensional reasoning.

Quantities associated with units of measurement are combined according to the fundamental principles of arithmetic and the laws of exponents.

$$\text{in} \times \text{in} = \text{in}^2$$
$$3\text{ in} \times 4\text{ in} = (3)(4)(\text{in})(\text{in}) = 12\text{ in}^2$$
$$\frac{6\text{ cm}^3}{2\text{ cm}} = \frac{6}{2}\text{ cm}^{3-1} = 3\text{ cm}^2$$
$$\frac{6\text{ ft} \times 3\text{ lb}}{9\text{ sec}} = 2\,\frac{\text{ft-lb}}{\text{sec}}$$
$$13\text{ g} - 12\text{ g} + 4\text{ g} = 5\text{ g}$$
$$2\text{ tons} + 5\text{ tons} + 100\text{ lb} = 7\text{ tons} + 100\text{ lb}$$

EXAMPLE 5-A

Find the volume of a right circular cylinder with diameter (d) equal to 6 in and height (h) equal to 14 in

Solution

Volume of cylinder, $V = \frac{\pi d^2}{4} \times h$

Thus,

$$V = \frac{\pi(6\text{ in})(6\text{ in})}{4} \times h = \pi\frac{(6)(6)(14)(\text{in})(\text{in})(\text{in})}{4}$$
$$= \left(\text{using the approximation } \frac{22}{7} \text{ as the value of } \pi\right)$$
$$= \frac{(22)\overset{9}{\cancel{(36)}}\overset{2}{\cancel{(14)}}}{\cancel{(7)}\cancel{(4)}} \times \text{in}^{1+1+1} = 396\text{ in}^3$$

EXAMPLE 5-B

Find the force, in pounds (lb), that will produce an acceleration of 18 ft/sec² when acting on a mass of 4,000 lb

Solution

This example is associated with Newton's Second Law of Motion, which can be expressed mathematically as

$$F = ma$$

where

$$m = \frac{\text{weight of body}}{\text{gravitational acceleration}} = \frac{w}{g}$$

g, acceleration resulting from the pull of gravity, will be taken as 32 ft/sec², a convenient approximation

Thus,

$$F = ma = \frac{w}{g} \times a = \frac{40 \times 10^2\text{ lb}}{32\,\frac{\text{ft}}{\text{sec}^2}} \times 18\,\frac{\text{ft}}{\text{sec}^2}$$
$$= \frac{40 \times 10^2 \times 18}{32} \times \frac{\frac{\text{lb ft}}{\text{sec}^2}}{\frac{\text{ft}}{\text{sec}^2}} = \frac{45 \times 10^2}{2} \times \frac{\text{lb}\cancel{\text{ft}}}{\cancel{\text{sec}^2}} \times \frac{\cancel{\text{sec}^2}}{\cancel{\text{ft}}} = 22.5 \times 10^2\text{ lb}$$

(Recall the rule for division of fractions: invert fraction in denominator and multiply, ... This applies to units of measurement as just demonstrated.)

EXERCISES 5-1

Carry out the indicated operations involving dimensional symbols.

1. $\text{cm} \times \text{cm} =$
2. $\text{cm}^2 \times \text{cm} =$
3. $\text{in.} \times \text{in.}^2 =$
4. $2\text{ in.} \times 3\text{ in.} =$
5. $3\,\frac{\text{cm}}{\text{sec}} + 5\,\frac{\text{cm}}{\text{sec}} =$
6. $\frac{\text{lb} \times \text{ft}}{\text{sec}} \div \frac{\text{lb} - \text{ft}}{\text{sec}} =$
7. $\frac{72{,}000\text{ ft-lb}}{\text{min}} \div \frac{36{,}000\text{ ft-lb}}{\text{min}} =$
8. $\text{lb}\left(\frac{\text{ft}}{\text{sec}}\right)^2 \div \frac{\text{ft}}{\text{sec}^2} =$
9. $\frac{\text{lb}\left(\frac{\text{ft}}{\text{sec}}\right)^2}{\text{ft}} =$
10. $\text{lb} \times \frac{\text{ft}^2}{\text{sec}^2} \times \left(\frac{\text{sec}}{\text{ft}}\right)^2 =$
11. $\frac{\text{ft-lb}}{\text{sec}} \div \frac{1}{\text{sec}} =$
12. $\text{cm}^3\left(\frac{\text{g}}{\text{cm}}\right)\left(\frac{\text{cm}}{\text{g}}\right) =$
13. $(\text{cm}^{-3})\left(\frac{\text{g}}{\text{cm}}\right)\left(\frac{\text{cm}}{\text{g}}\right) =$
14. $\frac{\text{g}}{\text{cm}^2} \times \text{cm}^3 \times \frac{\text{cm}}{\text{g}} =$
15. $\frac{\text{g}\left(\frac{\text{cm}}{\text{sec}^2}\right)}{\left(\frac{1}{\text{sec}}\right)^2} =$
16. $\frac{\text{cm}}{\text{sec}^2} \times \text{g cm} \times \left(\frac{\text{sec}}{\text{cm}}\right)^2 =$
17. $\frac{25\text{ cm} \times 10\text{ cm}}{\text{cm}^{-2}} =$
18. $\frac{1}{2}\,\text{g}\,\frac{\text{cm}^2}{\text{sec}^2} + \text{g}\,\frac{\text{cm}^2}{\text{sec}^2} =$
19. $\frac{72{,}000\text{ ft-lb}}{\text{min}} \div 550\,\frac{\text{ft-lb}}{\text{sec}} =$
20. $\text{g cm}^{-2} \times \text{g cm}^{-1} \times \text{sec}^{-2} =$
21. $\frac{\frac{\text{ft}}{\text{sec}}}{\frac{1}{\text{sec}^2}} \times \frac{\frac{\text{ft lb}}{\text{sec}}}{\frac{\text{ft}}{\text{sec}^2}} \times \frac{\frac{1}{\text{sec}}}{\frac{\text{lb}}{\frac{1}{\text{sec}}}} =$
22. 16 milliliters (ml) $-$ 7.7 ml $+$ 0.6 ml $=$
23. $612\,\frac{\text{in.-lb}}{\text{sec}} + 550\,\frac{\text{in.-lb}}{\text{sec}} =$
24. $100\,\frac{\text{cm}^2}{\frac{\text{cm}}{\text{sec}}} \div 10\,\frac{\text{cm}^2}{\frac{\text{cm}}{\text{sec}}} =$
25. $980\,\frac{\text{cm}}{\text{sec}^2} \div 32.16\,\frac{\text{ft}}{\text{sec}^2} =$

5-2 THE METRIC SYSTEM

The technician, while working with engineering data, will use the English units of measurement, referred to as foot-pound-second. Scientific mea-

surements, by and large, are based on the **centimeter-gram-second** (cgs) system, referred to also as the **metric system** Electrical units to some extent are more or less internationally standardized Notice the re emphasis of the three fundamental physical quantities length, mass, and time

From time to time it may become necessary to convert readings from one system to the other Also, units within the same system often are changed to other denominations, such as ounces to pounds, centimeters to meters, and so on

There will be occasion to use symbols representing extremely large quantities Likewise, development of modern sensitive instruments permit minute particles to become items of observation and record One ten thou sandth of an in (0 0001) is considered small, whereas, 0 000100 f (100 micro-farads, μf) is a comparatively large capacity rating Numbers associated with measurements must be considered in terms of the unit they represent

Most of the micro-kilo quantities that a technician will encounter are common shop talk in the metric system Anything under a thousandth is out of reach of mass production in the English system

Certain prefixes, used to complement metric-electrical units, along with their numerical equivalents, are given in Fig 5-1

The basic units of the cgs system are the **meter** (length) and the **gram** (mass), along with the universal **second** (time) Fig 5-2 lists basic metric units along with comparative auxiliary units

(In working with liquids, the unit milliliter (ml) is usually preferred over the unit cubic centimeter (cc), furthermore, in terms of solids and geometric forms, cubic centimeters will most often appear as cm^3 For example, under certain conditions, 1 liter of water will occupy a volume of 1000 cm^3 By and large, usage will depend on local practice and it is conceivable that the density of a certain material could be given by various sources as, 2 1 g/ml, 2 1 g/cc, or 2 1 g/cm^3)

5-3 ENGLISH UNITS

From the brief exposure to the contents of Fig 5-2, it becomes apparent that the subsidiary units are defined in multiples of 10 with reference to the basic units (gram meter) A megacycle, 1,000,000 cycles, really represents $10^6 \times$ the basic unit (cycle) Likewise, 1 milliamp (ma) is one-one thousandth of an ampere, or 1 ma $= 0\ 001$ a $= 1 \times 10^{-3}$ a

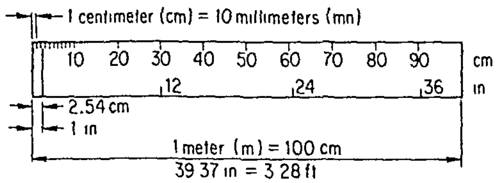
The rule for conversion to auxiliary units amounts to nothing more than selecting the appropriate multiple of 10 defined by the prefix (kilo—10^3, milli—10^{-3}) Notice that as the units tend to become smaller, more and more of them are needed to represent the equivalent larger unit (1 m $=$ 100 cm $=$ 1,000 mm)

EXAMPLE 5 C

Convert 150 cm to millimeters

Prefix	Numerical Equivalent	Scientific Notation
deca	10	10
hecto	100	1×10^{2}
kilo	1,000	1×10^{3}
mega	1,000,000	1×10^{6}
giga	1,000,000,000	1×10^{9}
terra	1,000,000,000,000	1×10^{12}
deci	0.1	1×10^{-1}
centi	0.01	1×10^{-2}
milli	0.001	1×10^{-3}
micro (μ)	0.000001	1×10^{-6}
milli micro (mμ)	0.000000001	1×10^{-9}
micro micro ($\mu\mu$)	0.000000000001	1×10^{-12}

Figure 5-1



Length

$1\ m = 100\ cm = 1000\ mm = 0.001$ kilometers (km)

$1\ m = 1 \times 10^{2}\ cm = 1 \times 10^{3}\ mm = 1 \times 10^{-3}\ km$

1 Angstrom $= 0.0000000001\ m = 1 \times 10^{-10}\ m = 1 \times 10^{-8}\ cm$

Mass

1 gram (g) = 1000 milligrams (mg) = 0.001 kilogram (kg)

$1\ g = 1 \times 10^{3}\ mg = 1 \times 10^{-3}\ kg$

$1 \times 10^{3}\ g = 1\ kg$

1 gamma = 1 micro gram (μg) = $0.000001\ g = 1 \times 10^{-6}\ g$

Time

1 second = 1000 millisecond = 1,000,000 microsecond, or

$1\ sec = 1 \times 10^{3}\ m\ sec = 1 \times 10^{6}\ \mu\ sec$

Volume

1 liter (l) = 1000 milliliters = 1000 cubic centimeters

$1\ l = 10^{3}\ ml = 10^{3} cc$

Figure 5-2

Length

1 foot (ft) = 12 inches (in.)

3 ft = 1 yard (yd)

1 rod = $16\frac{1}{2}$ ft

1 mile (mi) = 5280 ft = 320 rods

Mass

1 ounce = 437.5 grains

1 pound (lb) = 16 ounces (oz) = 7000 grains

1 ton = 2×10^{3} lb = 32×10^{3} oz

Volume

16 ounces = 1 pint (pt)

1 quart (qt) = 2 pints

1 gallon (gal) = 4 quarts = 8 pints

231 in.^{3} = 1 gal

Figure 5-3

Solution:

$$1 \text{ cm} = 10 \text{ mm}$$

or

1 cm is 10 times larger than 1 mm

Thus,

$$150 \text{ cm} = 150 \times 10 \text{ mm} = 15 \times 10^2 \text{ mm}$$

EXAMPLE 5-D:

Convert 600 mm to centimeters.

Solution:

$$1 \text{ mm} = 10^{-1} \text{ cm}$$

or

1 millimeter is $\frac{1}{10}$ of a centimeter

Therefore,

$$600 \text{ mm} = 600 \times 10^{-1} \text{ cm} = 6 \times 10^2 \times 10^{-1} \text{ cm}$$
$$= 6 \times 10 \text{ cm} = 60 \text{ cm}$$

EXAMPLE 5-E:

Convert 60 kg to grams and milligrams

Solution:

$$1 \text{ kg} = 10^3 \text{ g} = 10^6 \text{ mg}$$
$$60 \text{ kg} = 60 \times 10^3 \text{ g} = 60 \times 10^6 \text{ mg}$$

EXERCISES 5-2

Complete the table of conversions of units and auxiliary units (Ex 1-3 are illustrative)

1.	200 cm	2 m	2×10^3 mm
2.	10 μsec	10×10^{-6} sec	10×10^{-3} m sec
3.	6 kg	6×10^3 g	6×10^6 mg
4.	1 km	________ m	________ cm
5.	20 m	________ cm	________ mm
6.	90 sec	________ μsec	________ millisec
7.	________ a	6 5 ma	
8.	300 g	________ kg	________ mg
9.	________ mm	2×10^{-1} cm	________ m
10.	65×10^{-3} g	________ mg	
11.	0 2 μsec	________ millisec	________ sec

12.	6×10^3 w	________ kilowatts (kw)	________ milliwatts (mw)
13.	25×10^{-10} m	________ cm	________ Å
14.	45 gammas	________ mg	________ g
15.	3 Å	________ mm	________ cm
16.	________ Å	________ cm	12×10^{-7} mm
17.	62×10^{-3} mg	________ g	
18.	________ m	7.7×10^4 cm	7.7×10^5 ________
19.	6.2×10^3 g	6.2 ________	________ kg
20.	250 mm	2.5×10^2 ________	25 ________
21.	250 millisec	25×10^{-2} ________	25×10^4 ________
22.	3×10^6 ohms	________ megaohms	
23.	6×10^3 mm	6×10^{-3} ________	6 ________
24.	6×10^3 mg	6×10^{-3} ________	6 ________
25.	0.001 m	________ cm	________ km
26.	10 gal	________ pints	________ oz
27.	50 l	________ ml	________ cc
28.	3.5×10^2 ml	________ l	________ cc
29.	128 rods	________ yd	________ mi
30.	2.5 mi	________ ft	________ rods
31.	320 rods	1.0 ________	________ yd
32.	________ yd	4 mi	________ rods
33.	42.0×10^3 lb	________ tons	21 ________
34.	8.5×10^3 l	________ ml	
35.	3×10^2 yd	________ ft	________ in.
36.	1,055 in.^{3}	________ gal	
37.	1,000 m	________ mi	
38.	10^6 gammas	________ g	________ μg
39.	10^8 mi	________ Å	
40.	100 yd	________ m	

5-4 EQUIVALENT UNITS

A ratio is the indicated quotient of two identical quantities. It may be expressed as a fraction, decimal, percentage, or integer. Many pertinent technical concepts, such as specific gravity, are expressed as ratios. *Specific gravity is defined as the ratio of the density of a substance to the density of*

Un ts may be ea onged f need o ses

1 m 10^2 cm 10^3 mm

0 mm 1 cm

1 mm 10 cm 10^{3} m 10^{6} km

1 cm 0 mm 10^{2} m 10^{5} km

km 10^3 m 10^5 cm 10^6 mm

Figure 5 4

water In the metric system, this ratio is very convenient since the density of water is taken as 1 g/cm³

In most cases quantities expressed as ratios have the same units in /in

$$\frac{\frac{g}{cm^3}}{\frac{g}{cm^3}}, \quad \frac{\frac{lb}{ft^3}}{\frac{lb}{ft^3}}, \text{ etc}$$

At times it may become necessary to express equivalent quantities (dissimilar units) as ratios This is an approach used in the development of **conversion factors** *Conversion factors are numerical relationships that lead to equating units of one system in terms of another or to auxiliary units within the same system* For example 1 in = 2 54 cm Here, 2 54 is considered a conversion factor

The metric system is based on multiples of ten which makes it a decimal system The English system on the other hand is not Thus, there is no common factor that can be used to translate units of measurement interchangeably As a result conversion factors are developed empirically to approximate units of measurement, defining the same property

In this country, the standard units of measurement are kept at the Bureau of Standards in Washington Similar depositories exist throughout the world

Table 5 5 lists several common conversion factors that are used to transform data from the cgs system to the lb-ft-sec system All of these conversion factors may be expressed as ratios For example 2 54 cm/1 in = 1

This concept of expressing identical properties in the form of a ratio may prove to be a useful technique in converting related measurements

In order to change data from one system of units to another, a conversion factor or other relevant information is needed Several common English metric equivalents are reproduced in Fig 5 5 This table, although comprehensive is limited to those units most likely to be encountered by the technician From time to time unique conditions may warrant development of supplemental factors

EXAMPLE 5 F

Convert 10 in to an equivalent reading in centimeters

Solution

1 in = 2 54 cm

Length

1 inch = 2.54 centimeters
1 foot = 30.48 centimeters = 0.3048 meters
1 mile = 1.609 kilo meters
1 centimeter = 0.3937 inches
1 meter = 39.37 inches = 3.2808 feet
1 kilometer = 0.6214 miles

Volume

1 quart = 0.9463 liters = 946.3 cm^3 (ml)
1 fluid ounce = 29.57 milliliter
1 gallon = 3.785 liters
1 liter = 1.057 quarts
1 liter = 61.025 in^3
1 cubic centimeter = 61.025×10^{-3} in^3
1 cubic centimeter = 3.38×10^{-2} *ounces = 1 milli liter*

Mass

1 pound = 16 ounces (The pound or ounce is a unit of weight that is *referred to as avoirdupois.*)
1 pound (avoirdupois) = 16 ounces
1 pound = 453.6 grams (454 grams)
1 ounce = 28.35 grams
1 ton (2000 lb) = 9.07×10^2 kilo grams
1 kilo gram = 2.205 pounds (rounded to 2.2 lb)
1 gram = 2.2×10^{-3} pounds

Temperature

Fahrenheit	°F	Centigrade °C
Freezing point of water	32°	0°
Boiling point of water	212°	100°

$$°F = \tfrac{9}{5}°C + 32° \qquad °C = \tfrac{5}{9}(°F - 32°)$$

Figure 5-5

Thus;

$$10 \text{ in.} = 10 \times 2.54 \text{ cm} = 25.4 \text{ cm}$$

EXAMPLE 5-G:

Convert 18 cm to inches.

Solution:

$$1 \text{ cm} = 0.3937 \text{ in.}$$

Hence;

$$18 \text{ cm} = 18 \times 0.3937 \text{ in.} = 7.087 \text{ in.}$$

(Given data and intended use will determine the final form of the answer. In the example, the conversion was rounded to 3 places.)

EXAMPLE 5-H:

How many ounces are there in a 500-ml solution?

Solution:

$$1 \text{ ml} = 3.38 \times 10^{-2} \text{ oz}$$

Therefore,

$$500 \text{ ml} = 500 \times 3.38 \times 10^{-2} \text{ oz}$$
$$= (5 \times 3.38)(10^2 \times 10^{-2}) = 16.9 \text{ oz}$$

EXAMPLE 5-1·

Find the equivalent of 980 g in pounds

Solution·

$$1\ g = 2\,2 \times 10^{-3}\ lb$$
$$980\ g = 980 \times 2\,2 \times 10^{-3}\ lb$$
$$980\ g = (9\,8 \times 2\,2) \times (10^{2} \times 10^{\ 3})\ lb$$
$$= 21\,56 \times 10^{\ 1}\ lb = 2\,156\ lb = 2\,16\ lb$$

EXERCISES 5-3

Make the following conversions

1. 36 in to ____ m

2. 100 m to ____ yd

3. 16 qt to ____ ml ____ l

4. 12 l to ____ gal, ____ oz

5. 12×10^{3} l to ____ gal, ____ oz

6. 20°C to ____ °F

7. 20°F to ____ °C

8. 30×10^{3} kg to ____ lb, ____ tons

9. 1,000 m to ____ yd, ____ miles

10. $62\,14 \times 10^{\ 2}$ miles to ____ km

11. 10×10^{3} m to ____ yd, ____ miles, ____ km

12. $0\,1 \times 10^{-3}$ in to ____ mm

13. 100 yd to ____ m

14. 100 ft to ____ m

15. 40 km to ____ miles, ____ m

16. $39\,37 \times 10^{2}$ in to ____ cm, ____ m

17. $30\,48 \times 10^{2}$ cm to ____ ft

18. $30\,48 \times 10^{\ 2}$ cm to ____ ft

19. $37\,85 \times 10^{-3}$ l to ____ gal, ____ ml

20. $94\,63 \times 10^{4}$ cc to ____ qt

21. $122\,05\ in^{3}$ to ____ l

22. 122 05 l to ____ in^{3}

23. 19×10^{3} ml to ____ oz, ____ cc

24. 54 gal to ____ l, ____ in^{3}

25. 54 l to ______ gal, ______ qt

26. 100 g to ______ oz, ______ lb

27. 1×10^{-1} kg to ______ oz ______ lb

28. 18×10^{3} kg to ______ lb

29. 20 tons to ______ kg

30. 1×10^{-3} g to ______ oz, ______ mg

31. 3×10^{-3} g to ______ oz, ______ mg

32. 3×10^{6} μg to ______ oz, ______ mg

33. 16 A° to ______ in., ______ mm

34. 24 oz to ______ g, ______ mg

35. 72°C to ______ °F

36. 72°F to ______ °C

37. 50°C to ______ °F

38. 122°F to ______ °C

39. 300 °F to ______ °C

40. 500°C to ______ °F

41. 144 in.^{2} to ______ ft^2, ______ cm^2

42. 1.0×10^{4} cm^2 to ______ m^2, ______ in.^{2}

43. 1,728 in.^{3} to ______ ft^3, ______ m^3

44. 9 ft^2 to ______ yd^2, ______ m^2

45. 27 ft^3 to ______ in.^{3} ______ yd^3

5-5 FORMULAS

Many things occur in nature (natural phenomena) that later lead to the development of scientific laws (theory) that define the phenomena. The pull of gravity was present the day the earth was created but wasn't discovered supposedly until Sir Isaac Newton saw an apple fall. Electricity sparked with the first thunderstorm. Tons of pitchblende and a decade of dedication led to the discovery of a few milligrams of radium. Characteristics of human beings and other statistical behavior can be measured with reference to the probability curve, which in turn is developed around the number, $e = 2.71828$, the base of the natural logarithms.

Many scientific laws and engineering developments can be traced back first to an occurrence, followed later by theoretical analysis of the pattern of behavior. These relationships are then combined according to some mathematical format and called **formulas.**

Formulas or equations are a method of expressing, in symbols, various scientific-engineering relationships. Formulas are developed through a

combination of procedures involving definitions, experiments, or dimensional analysis Chemical formulas, for example, used to represent the composition of chemical compounds, are usually established by experiment

A formula can also be referred to as a statement of equality involving the behavior of several quantities $F = ma$, $S = MC/I$, $E = IR$, $D = M/V$, $V = \pi r^2 h$, and so on If the elements in a formula are not separated by a plus (+) or minus (−) sign, it is understood that the factors are combined by multiplication $ma = m \times a$, $IR = I \times R$, $\pi r^2 h = \pi \times r^2 \times h$, and so on

5 6 TRANSPOSING TERMS

A formula defines a concept in terms of various factors or components The volume of a sphere is defined mathematically as $V = \frac{4}{3}\pi r^3$, where r is the radius In its present form, the equation can only be used to find the volume, given the radius To limit the use of this formula (or any other formula for that matter) strictly to this one function, however, is to lose sight of the true mathematical connotation This equation, $V = \frac{4}{3}\pi r^3$, defines the volume of a sphere in terms of its radius Thus, for a given radius there is only one corresponding volume And for a given volume there is only one corresponding radius From all of this, it would seem that there exits a condition leading to an equation involving the radius in terms of the volume

Actually, the properties of any formula may be interchanged That is, any component of an equation (no matter how complex the equation) may be obtained in terms of the other elements The quantities of a formula may be re-arranged or transposed, to fit a need, if this development is carried out in accordance with established mathematical (dimensional) procedures

To accomplish this purpose, some guidelines will have to be established The following illustrations are pointed toward this goal

Whenever the subject of an equation or formula enters a technical discussion, it carries along with it the all too familiar cry, **Make sure that the equation balances**, a point to bear in mind **Balance the equation, mathematically and dimensionally** By and large these two terms are synonmous when dealing with formulas

A 5 g weight and a 2-g weight are of the same mass as a 4-g weight plus a 2-g weight and a 1-g weight As a result, the scale in Fig 5-6 will balance If the 2-g weight were removed from the left pan, the equal balance would no longer be maintained Thus, to maintain balance, an equivalent amount of weight would have to be removed from the right pan The scale would then take on the appearance of Fig 5-7

Thus, $5\text{ g} + 2\text{ g} - 2\text{ g} = 4\text{ g} + 2\text{ g} + 1\text{ g} - 2\text{ g}$ leading further to $5\text{ g} = 4\text{ g} + 1\text{ g}$

Suppose that 6 g were added to the right pan of Fig 5-7 Certainly the scale would tip to the right, indicating unequal balance (Fig 5-8)

Again, to attain equal balance, 6 g would have to be added to the left (Fig 5-9)

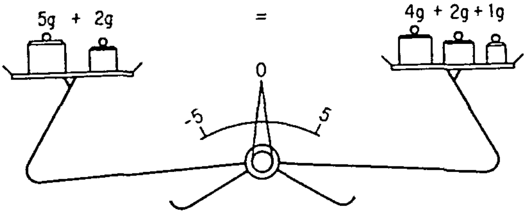


Figure 5-6

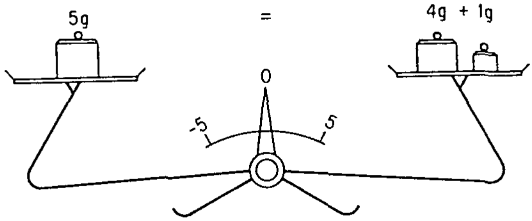


Figure 5-7

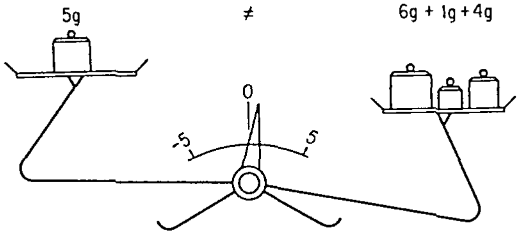


Figure 5-8

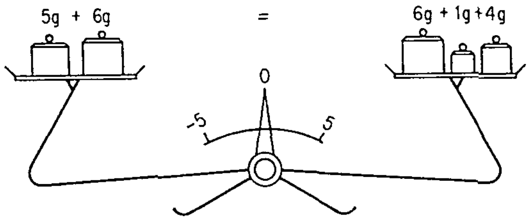


Figure 5-9

Thus, $5\text{ g} + 6\text{ g} = 4\text{ g} + 1\text{ g} + 6\text{ g}$, or $11\text{ g} = 11\text{ g}$.

If the weights on the left pan in Fig. 5-9 were tripled (multiplied by 3), the weights on the right pan would have to be tripled to maintain balance.

$$3(5\text{ g} + 6\text{ g}) = 3(4\text{ g} + 1\text{ g} + 6\text{ g})$$
$$15\text{ g} + 18\text{ g} = 12\text{ g} + 3\text{ g} + 18\text{ g}$$
$$33\text{ g} = 33\text{ g}$$

Similarly, if the weights on the right in Fig. 5-9, were halved (divided by 2), the weight on the left would be subjected to the same consideration if the meaning of the equality sign (=) were to be maintained.

$$\frac{5\text{ g} + 6\text{ g}}{2} = \frac{4\text{ g} + 1\text{ g} + 6\text{ g}}{2}$$
$$\frac{11\text{ g}}{2} = \frac{11\text{ g}}{2}$$

The symbol, =, by definition states that quantities separated by this symbol are equal For example, $6 + 3 = 4 + 5$, $E = IR$,

Axiom *Both sides of an equation may be increased or decreased multiplied or divided by the same property or quantity without destroying the equality*

The ability to manage equations or formula is of utmost importance in the work of the technician This axiom allows for transposing terms from one side of the equation to the other This leads further to a method of solving mathematically any one of the elements included in an equation or formula Sometimes several steps may be required to complete the process

EXAMPLE 5 J

Centrifugal force (CF) — mv^2/r, where m is mass in grams, v velocity in meters per second, and r the radius of the path, measured in meters

$$CF = \frac{mv^2}{r} \text{ can be represented dimensionally as } \frac{\text{g m}}{\text{sec}^2} = \frac{\frac{\text{g m}^2}{\text{sec}^2}}{\text{m}}$$

Solve for r, the radius or express r in terms of the other components

Solution

Step 1 Multiply both sides of the formula by r

$$r \times CF = \frac{mv^2}{\cancel{r}} \times \cancel{r} \text{ (} r \text{ on the right will cancel)}$$

Thus,

$$r \times CF = mv^2$$

Step 2 Divide both sides by CF

$$\frac{r \times \cancel{CF}}{\cancel{CF}} = \frac{mv^2}{CF} \left(\frac{CF}{CF} = 1 \text{ or } \frac{CF}{CF} \text{ on left will cancel out}\right)$$

Thus,

$$r = \frac{mv^2}{CF}$$

This can be checked dimensionally, recalling that r is measured in meters

(Dimensionally) $$r = \frac{\text{g} \times \frac{\text{m}^2}{\text{sec}^2}}{\text{g} \frac{\text{m}}{\text{sec}^2}} = \frac{\cancel{\text{g}} \times \frac{\text{m}^2}{\text{sec}^2}}{\cancel{\text{g}} \times \frac{\text{m}}{\text{sec}^2}}$$

$$= \frac{\frac{\text{m}^2}{\text{sec}^2}}{\frac{\text{m}}{\text{sec}^2}} = \frac{\text{m}^2}{\cancel{\text{sec}^2}} \times \frac{\cancel{\text{sec}^2}}{\text{m}} = \frac{\text{m}^2}{\text{m}} = \text{m}$$

Thus, $r = mv^2/CF$ is also dimensionally correct.

EXAMPLE 5-K:

Solve for the depth, h, given the formula $F = AhD$. F is the total force of a liquid, with density D, acting on the bottom of a container of area A. The height of the liquid in the container is represented by h. F is the force (lb), A is the area of the bottom of the container (ft^2), and D, the density of the liquid (lb/ft^3).

Solution:

$$F = AhD \text{ (dimensionally); } \mathrm{lb} = \mathrm{ft}^2 \times \mathrm{ft} \times \mathrm{lb/ft}^3$$

Divide both sides of the formula by the product $A \times D$.

$$\frac{F}{AD} = \frac{A \times Dh}{A \times D}$$

and

$$\frac{F}{A \times D} = h, \quad \text{or} \quad h = \frac{F}{AD}$$

Dimensional check: h must be in units of feet.

$$h = \frac{\mathrm{lb}}{\mathrm{ft}^2 \times \frac{\mathrm{lb}}{\mathrm{ft}^3}} = \frac{\mathrm{lb}}{\frac{\mathrm{lb}}{\mathrm{ft}}} = \mathrm{lb} \times \frac{\mathrm{ft}}{\mathrm{lb}} = \mathrm{ft}$$

EXAMPLE 5-L:

$X_c = 1/2\pi fC$ is the formula that expresses capacitive reactance, X_c, in ohms; where f is the frequency in cycles per second (cps) and C is the capacitance in farads. Find f, if $X_c = 200$ ohms and $C = 12.5 \times 10^{-6} f$.

Solution:

Rewrite $X_c = 1/2\pi fC$ in terms of f (multiply both sides by f).

$$f \times X_c = \frac{1 \times f}{2\pi fC} = \frac{1}{2\pi C}$$

Next, divide both sides by X_c.

$$\frac{f \times X_c}{X_c} = \frac{\frac{1}{2\pi C}}{X_c}$$

Thus,

$$f = \frac{1}{2\pi C X_c}$$

To complete the problem, substitute numerical quantities for respective symbols.

$$f = \frac{1}{2\pi \times 12.5 \times 10^{-6} \times 200} = \frac{1}{4\pi \times 12.5 \times 10^{-4}} = \frac{10^4}{50\pi} = 63.7 \text{ cps}$$

EXERCISES 5-4

In exercises 1-10 solve for the indicated elements (check for dimensiona accuracy)

Formula	Solve for
1. $V = \pi r^2 h$ V, volume, cm^3 r, radius, cm h, height, cm	h
2. $A = \frac{bh}{2}$ A, area of triangle, in.^{2} b, base, in h, altitude, in	b and h
3. $F = \frac{AhD}{2}$ F, *force*, g A, surface area, cm^2 h, depth, cm D, density, g/cm^3	A and h
4. $S = \frac{1}{2} at^2$ S, distance, ft a, acceleration, ft/sec^2 t, time in seconds	a and t^2
5. $V_a = \frac{V_i + V_f}{2}$ V_a, average velocity, cm/sec V_i, initial velocity, cm/sec V_f, final velocity, cm/sec	V_i and V_f
6. $S = \frac{1}{2} a(2t - 1)$ S, distance, cm a, acceleration, cm/sec^2 t, time, sec	a and t
7. $S = \frac{Mc}{I}$ S, stress, lb/in.^{2} M, moment, lb-in c, distance from neutral axis, in I, moment of inertia, in.^{4}	c and I

8. $MA = \dfrac{2C_1}{C_1 - C_2}$ $\qquad C_2$

MA, mechanical advantage (ratio—no units)
C_1, circumference of large wheel, ft
C_2, circumference of small wheel, ft

9. $V = \dfrac{\pi}{4} h(D^2 - d^2)$ $\qquad h$

V, volume of pipe, in.^{3}
h, height or length of pipe, in.
D, outside diameter, in.
d, inside diameter, in.

10. $V = 2\pi rn$ $\qquad r$ and n

V, linear velocity of a point moving on a curve, ft/sec
r, radius of circle, ft
n, revolutions per minute, rpm

Solve as indicated.

	Formula	Solve for
11.	$I = \dfrac{E - e}{R}$	E, e, and R
12.	$T = \dfrac{1}{a} + t$	a and t
13.	$W = \dfrac{2PR_1}{R_1 - R_2}$	R_2
14.	$\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$	r_1 and r_2
15.	$I = \dfrac{E}{r + \dfrac{R}{n}}$	r, R, and n
16.	$H = \dfrac{N + 2}{P}$	P and N

Exercises 17-26 refer back to the formulas of exercises 1-10, respectively.

	Formula with data	Solve for
17.	$V = \pi r^2 h$ $V = 125.6\text{ cm}^3$, $r = 2\text{ cm}$, $\pi = 3.14$	h
18.	$A = \dfrac{bh}{2}$ $A = 144\text{ in.}^2$, $b = 24\text{ in.}$	b
19.	$F = \dfrac{AhD}{2}$ $F = 1\text{ kg}$, $A = 64\text{ cm}^2$, $D = 4\text{g/cm}^3$	h

20. $S = \frac{1}{2}at^2$ $\qquad$ a

$S = 600$ m, $t = 5$ sec

21. $V_a = \frac{V_i + V_f}{2}$ $\qquad$ V_f

$V_a = 60$ ft/sec, $V_i = 22$ ft/sec

22. $S = \frac{1}{2}a(2t - 1)$ $\qquad$ t

$S = 1{,}500$ m, $a = 32$ ft/sec^2

23. $S = \frac{Mc}{I}$ $\qquad$ c

$S = 12 \times 10^2$ lb/in^2, $I = 900$ in^4, $M = 20 \times 10^4$ lb-in

24. $MA = \frac{2C_1}{C_1 - C_2}$ $\qquad$ C_2

$MA = 4$, diameter of large wheel $= 4$ ft

25. $V = \frac{\pi}{4}h(D^2 - d^2)$ $\qquad$ h

$V = 4$ in^3, $D = 3$ in, $d = 2.5$ in

26. $V = 2\pi rn$ $\qquad$ r

$V = 8\pi$in/sec, $n = 10$ revolutions/sec

Solve for the indicated term

Formula	Term
27. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ $R = \frac{50}{7}$, $R_1 = 25$, $R = 15$ (Resistance in parallel)	R_3
28. $R = \frac{R_1 \times R_2}{R_1 + R_2}$ $R = 10$, $R_1 = 30$	R_2

5-7 DERIVATION OF CONVERSION FACTORS

Dimensional conversions up to this point have involved single-unit expressions such as the centimeter, inch, liter, and kilogram. Many technical phenomena, however, are recorded in multiunits, such as lb/in^2, g/cm^3, gal/min, ft-lb/sec^2.

The procedure for converting this type of data from one system to another may require several steps. Often this may require the development of a new formula or unique equation based on definitions.

EXAMPLE 5-M

Derive a formula that will convert tons to ounces (and ounces to tons).

Solution:

Step 1. Find a known common unit for the properties involved, one that can be expressed as an equivalency.

In this example the common unit for the ounce and ton is the pound.

$$16 \text{ oz} = 1 \text{ lb}$$

$$2{,}000 \text{ lb} = 1 \text{ ton}$$

(16 and 2,000 may be viewed as the equivalency factors)

Step 2. Develop a dimensional equation that will express one unit in terms of the other unit. The importance of unit selection must be stressed. It is almost mandatory to use comparative units that can be expressed as a ratio equal to 1.

$$\frac{16 \text{ oz}}{1 \text{ lb}} = 1, \text{ or } \frac{1 \text{ lb}}{16 \text{ oz}} = 1$$

$$\frac{2{,}000 \text{ lb}}{1 \text{ ton}} = 1, \text{ or } \frac{1 \text{ ton}}{2{,}000 \text{ lb}} = 1$$

Occasionally, in the first attempt to develop a formula for conversion, the order of appearance of the units in the fraction may be unfavorable. This is no great setback, for the order may be interchanged and the reciprocals will also be equal to 1.

$$\frac{16 \text{ oz}}{1 \text{ lb}} \text{ and } \frac{1 \text{ lb}}{16 \text{ oz}} \text{ are reciprocals}$$

Similarly,

$$\frac{2{,}000 \text{ lb}}{1 \text{ ton}} \text{ is the reciprocal of } \frac{1 \text{ ton}}{2{,}000 \text{ lb}}$$

To implement step 2, the following expression is developed:

$$1 \text{ ton} = 1 \text{ ton} \times \frac{\text{lb}}{\text{ton}} \times \frac{\text{oz}}{\text{lb}}$$

Obviously, 1 ton is not equal to 1 oz. Basically, the problem resolves itself to finding a factor that can be used to convert ton weights to ounces. Mathematically, this can be expressed as

$$\text{ton} = F_c \times \text{oz}$$

where F_c is the factor of conversion.

Step 3. After the dimensional expression is developed, numerical factors associated with the respective set of units are introduced. This is followed by appropriate arithmetic considerations that will lead to a balanced equation or formula (numerically and dimensionally).

Thus,

$$1 \text{ ton} = 1 \text{ ton} \times \frac{2{,}000 \text{ lb}}{1 \text{ ton}} \times \frac{16 \text{ oz}}{1 \text{ lb}} = 32{,}000 \text{ oz}$$

This is justified mathematically on the basis that

$$\frac{2{,}000 \text{ lb}}{1 \text{ ton}} = 1 \text{ and } \frac{16 \text{ oz}}{1 \text{ lb}} = 1$$

or,

$$1 \text{ ton} = 1 \text{ ton} \times 1 \times 1$$

Therefore,

$$1 \text{ ton} = 32{,}000 \text{ oz} = 3.2 \times 10^4 \text{ oz}$$

or,

$$1 \text{ oz} = \frac{1 \text{ ton}}{3.2 \times 10^4} = 3.12 \times 10^{-5} \text{ ton}$$

$F_c = 3.2 \times 10^4$ in the first equation, whereas

$F_c = 3.12 \times 10^{-5}$ in the second equation

EXAMPLE 5-Na

What is the weight, in ounces, of 22 tons of steel

Solution

$$1 \text{ ton} = 3.2 \times 10^4 \text{ oz}$$

Thus,

$$22 \text{ ton} = 22 \times 3.2 \times 10^4 = 7.0 \times 10^5 \text{ oz}$$

EXAMPLE 5-Nb

Find the weight, in tons, of 64×10^4 oz of sea water

Solution

$$1 \text{ oz} = 3.12 \times 10^{-5} \text{ ton}$$

Thus,

$$64 \times 10^4 \text{ oz} = 64 \times 10^4 \times 3.12 \times 10^{-5} \text{ ton} = 20 \text{ ton}$$

Note The known equivalencies, 1 lb = 16 oz and 1 ton = 2,000 lb, provided the mathematical justification that led to a procedure for converting weight in tons to an equivalent expression involving ounces Once determined, the factor of conversion, F_c, more or less summarizes the arithmetic details of the intermediate steps Every contributing item or factor is compounded within the symbol F_c, thus, there is no need for further concern about dimensional balance when the symbol is used

EXAMPLE 5 O

Develop a numerical factor that will convert gallons to cubic feet

Solution:

$$\text{gal} = F_c \times \text{ft}^3$$

Step 1.

(Common Units)

$$231 \text{ in.}^3 = 1 \text{ gal}$$
$$1{,}728 \text{ in.}^3 = 1 \text{ ft}^3$$

Step 2.

(Dimensional Equation)

$$1 \text{ gal} = 1 \text{ gal} \times \frac{\text{in.}^3}{\text{gal}} \times \frac{\text{ft}^3}{\text{in.}^3}$$

Step 3.

(Numerical Factors)

$$\text{gal} = F_c \times \text{ft}^3$$
$$1 \text{ gal} = 1 \text{ gal} \times \frac{231 \text{ in.}^3}{1 \text{ gal}} \times \frac{1 \text{ ft}^3}{1{,}728 \text{ in.}^3} = 0.134 \text{ ft}^3$$

Thus,

$$F_c = 0.134$$

Furthermore,

$$1 \text{ gal} = 0.134 \text{ ft}^3$$

or

$$1 \text{ ft}^3 = \frac{1}{0.134} \text{ gal} = 7.5 \text{ gal}$$

EXAMPLE 5-P:

Express $908 \frac{\text{g}}{\text{cm}^2}$ in terms of $\frac{\text{lb}}{\text{ft}^2}$

Solution:

(Common Units)

$$1 \text{ in.} = 2.54 \text{ cm}$$
$$1 \text{ in.}^2 = (2.54 \text{ cm})^2 = 6.45 \text{ cm}^2$$
$$1 \text{ ft} = 12 \text{ in.}$$
$$1 \text{ ft}^2 = 144 \text{ in.}^2$$
$$1 \text{ lb} = 454 \text{ g}$$

(Dimensional Equation)

$$\frac{\text{g}}{\text{cm}^2} = \frac{\text{g}}{\text{cm}^2} \times \frac{\text{lb}}{\text{g}} \times \frac{\text{cm}^2}{\text{in.}^2} \times \frac{\text{in.}^2}{\text{ft}^2}$$

$$\frac{g}{cm^2} = F_c \times \frac{lb}{ft^2}$$

$$\frac{g}{cm^2} = \frac{g}{cm^2} \times \frac{1\ lb}{454\ g} \times \frac{6\ 45\ cm^2}{1\ in^2} \times \frac{144\ in^2}{1\ ft^2}$$

Thus,

$$F_c = \frac{1 \times 6\ 45 \times 144}{454} = 2\ 04$$

Therefore,

$$\frac{g}{cm^2} = 2\ 04 \frac{lb}{ft^2}$$

or

$$\frac{lb}{ft^2} = \frac{1}{2\ 04} \frac{g}{cm^2} = 0\ 49 \frac{g}{cm^2}$$

Hence,

$$908 \frac{g}{cm^2} = 908 \times 2\ 04 \frac{lb}{ft^2} = 1{,}852 \frac{lb}{ft^2}$$

EXERCISES 5 5

In exercises 1-10, develop an equation that will lead to the appropriate conversion

1. cubic inches to cubic yards

2. square rods to square miles

3. gram centimeters to pound-feet

4. ounces to kilograms

5. miles per hour to feet per second

6. feet per minute to centimeters per second

7. centimeters per second to miles per hour

8. amperes per square inch to amperes per square centimeter

9. feet per second to kilometers per hour

10. feet per second to meters per second

11. Which of these two readings indicates fastest time 100 yd in 9 4 sec or 100 m in 10 2 sec?

12. How many tons of sea water, containing $2\ 45 \times 10^{-10}$ percent gold, are needed to yield 1 g of gold?

13. A 900 ml solution contains 25 per cent alcohol How much alcohol is there in 30 oz of this solution?

14. The density of water is approximately 62 4 lb/ft³ Find the density of water in terms of grams per cubic centimeter

15. One liter of oxygen weighs 1.429 g. Find the volume of 16 oz of oxygen.

16. The tensile strength of aluminum is approximately 35,000 $lb/in.^2$ Express this in terms of kilograms per square centimeter.

17. 1 grain $= 64.8 \times 10^{-3}$ g. Find the equivalent of 1 grain in ounces.

18. The coefficient of linear expansion of copper is 0.7×10^{-6} increase in unit length per degree Centigrade. What is the coefficient of linear expansion in unit length per degree Fahrenheit?

19. Express 1.9 $grains/ft^3$ as grains per cubic meter.

20. The velocity of sound in air is approximately 1,087 ft/sec. Find the velocity of sound in terms of meters per second.

21. The velocity of sound through glass is taken as 16.5×10^3 ft/sec. Express this as equivalent centimeters per second.

22. Record an atmospheric pressure of 14.7 $lb/in.^2$ in equivalent grams per square centimeter.

23. Density of a substance is defined as the ratio of the mass of the substance to the volume it occupies, or, density = mass/volume. In the metric system, this is written as grams per cubic centimeter and in the English system, as pounds per cubic foot. If the density bronze is 8.8 g/cm^3, how many cubic centimeters would 88 g occupy?

24. Two lb of silver occupy a volume of 86.46 cm^3. Find the density of silver in grams per cubic centimeter.

25. If the density of gold is 19.3 g/cm^3, how much will 1 $in.^3$ of gold weigh?

REVIEW EXERCISES 5-6

Combine units according to the indicated operations (Ex. 1-5).

1. $\frac{\text{cm}}{\text{lb}} \times \frac{\text{g}}{\text{cm}} \times \text{lb} =$

2. $\frac{\text{lb}}{\text{ft}^3} \times \frac{1\ \text{lb}}{454\ \text{g}} \times \text{ft}^2 =$

3. $\frac{\text{in.}}{\text{sec}} \times \frac{\text{sec}^2}{\text{ft}^2} \div (\text{in.})^2 =$

4. $\frac{\text{oz}}{\text{g}} \times \frac{\text{lb}}{\text{cm}^2} \times \frac{\text{g(cm)}^2}{\text{oz}} =$

5. $(\text{g cm}^{-2}) \times \left(\frac{\text{cm}}{\text{g}}\right)^2 \div \left(\frac{1}{\text{cm}}\right)^3 =$

Complete the conversions (Ex. 6-15).

6. 40 grains = ______ oz

7. 16 ft^3, ______ m^3, ______ $in.^3$

8. 3 miles, ______ km

9. 100 mm^2, ______ cm^2, ______ $in.^2$

10. 100 decimeters (dm), ______ m, ______ cm

11. 960 grains, ______ g, ______ oz

12. ______ mm^3, ______ in^3, 1 cm^3

13. 2 ______, 908 ______, ______ oz

14. 1.1012 l, ______ in^3

15. 3,500 ______, ______ oz, 0 5 ______, 227 ______

Find the factor of conversion (Ex 16-20)

16. cubic inches to cubic meters

17. cubic inches per second to cubic meters per minute

18. milligrams per milliliter to ounces per cubic inch

19. $\frac{\text{ton}}{\text{min}}$ to $\frac{\text{kg}}{\text{min}}$

20. $\frac{\text{km}}{\text{min}}$ to $\frac{\text{miles}}{\text{hr}}$

In problems 21-36, derive a formula or equation that will balance dimensionally The factors should be arranged arithmetically to provide the required dimensional relationship among the given physical properties

21. The modulus of elasticity is given in units of pounds per square inch This formula provides a means of comparing the elasticity of various materials and involves, stress (s) in pounds per square inch and strain (e), which is a ratio (no units) Derive a formula for E in terms of stress and strain

22. Work (W) is expressed foot-pounds Derive a formula for work given the distance (s) in feet and force (f) in pounds

23. Kinetic energy is measured in terms of joules, a unit defined as kg-m^2/sec^2 Develop a dimensional equation for kinetic energy (KE) involving mass (m) in kilograms and velocity (v) in meters per second

24. Potential energy (PE), also measured in joules, involves mass, height (h), in meters, and the pull of gravity (g)m/sec^2 Develop a formula for PE

25. Find an equation for momentum (M), kg-m/sec, which is defined in terms of mass and velocity

26. Centrifugal force (CF) is a force acting away from a center along a path whose radius (r) is measured in meters Other components are mass and velocity Derive an equation for CF whose units are kg-m/sec^2

27. Find the equivalent, in grains of 1 oz of gold

28. What is the volume of 100 g of water whose density was measured as $97\,489 \times 10^{-2}$ g/ml?

29. Power (P) is defined as time rate of doing work ($P = w/t$) with units ft-lb/sec or kg-cm/sec Find the relationship for t (time in seconds) in terms of power (P) and work (W)

30. The total force (F) acting on the bottom of a container, with liquids, is

defined in pound units. The other factors associated with this force are the density in pounds per cubic feet of the liquid in the container, the area (a) of the bottom of the container in square feet, and the height (h) of the liquid in the container in feet. Develop a formula for F, using the contributing components.

Solve for the indicated element. (Ex. 31–40).

Formula	Solve for
31. Magnifying power of telescope.	
$MP = \dfrac{25 \text{ cm} \times L}{f_e \times f_o}$	f_e
$MP = 100$, $L = 12$ cm, $f_o = 0.75$ cm	
32. $S = \dfrac{Mc}{I}$	M(lb-ft)
$S = 6 \times 10^3$ lb/in.^{2}, $I = 500$ in.,^{4} $c = 5$ in.	
33. Linear expansion	
$L = \alpha l(T_2 - T_1)$	T_1
$L = 0.156$ in., $\alpha = 6.5 \times 10^{-6}$ (coefficient of linear expansion carries no units)	
$T_2 = 100°$F (final temperature)	
$l = 20$ ft (original length)	
34. Thermal stress	
$S = \alpha E(T_2 - T_1)$	S
Find the stress developed in problem 33 owing to change in temperature.	
$E = 30 \times 10^6$ lb/in.^{2}	
35. $r = \dfrac{m}{d-L} - \dfrac{m}{d+L}$	m,
36. Maximum deflection for a simple beam with load, W, in center	
$y = \dfrac{WL^3}{48\ EI}$	W(lb)
$y = 0.250$ in., $L = 10$ ft	
$E = 30 \times 10^6$ lb/in.^{2}, $I = 2 \times 10^2$ in.^{4}	
(check dimensionally before substituting)	
37. Maximum deflection for a cantilever beam with uniform loading W.	
$y = \dfrac{WL^3}{8\ EI}$	I
$y = 0.764$ in., $W = 6.0 \times 10^3$ lb	
$L = 24$ ft, $E = 30 \times 10^6$ lb/in.^{2}	
38. $S = \dfrac{Mc}{I}$	S
Find the maximum stress developed in the beam in problem 37.	

$c = \frac{15}{2}$ in , $M = 6 \times 10^3$ lb $\times \frac{24 \times 12}{2}$ in

39. $\frac{a+b}{b} = \frac{c+d}{d}$ b and d

40. $T = r(s + r)$ s

CHAPTER 6

Ratio—Proportion —Variation

A **ratio** has been defined as the indicated quotient of two related properties. *A statement of equality between two ratios is called a proportion.*

$$\frac{a}{b} = \frac{c}{d} \qquad \frac{3}{5} = \frac{21}{35} \quad (b \neq 0, d \neq 0)$$

Perhaps the most familiar examples of proportions are found in the historic gas laws.

$V_1/V_2 = T_1/T_2$, where V_1 is the original volume (cm^3 or ft^3), and V_2 is the new volume resulting from a change in temperature from T_1 to T_2.

Another form of the same proportion, after transposing, becomes:

$$\frac{V_1}{T_1} = \frac{V_2}{T_2}$$

The ratio V_1/T_1 (or V_2/T_2) is made up of unlike units; however, the equation (proportion) remains dimensionally sound.

$V_1/T_1 = V_2/T_2$ or $cm^3/°K = cm^3/°K$, where the temperature is given in absolute units, or Kelvin units: $°K = 273° + °C$.

The last expression again serves as an example of how mathematical analysis and dimensional analysis complement each other.

$$\frac{V_1}{V_2} = \frac{T_1}{T_2} \text{ or } \frac{cm^3}{cm^3} = \frac{°K}{°K} \text{ and } \frac{V_1}{T_1} = \frac{V_2}{T_2} \text{ or } \frac{cm^3}{°K} = \frac{cm^3}{°K}$$

After transposing further, the same relationship can be expressed as $V_2 = V_1/T_1 \times T_2$, which states that the volume, V_2, varies with the temperature, T_2. If T_2 increases, then V_2 will increase. Likewise, if the temperature decreases, the volume will decrease. Thus, the value of V_2 will vary whether V_1/T_1 is multiplied by a small number or a large number; or, in terms of variation, the volume of gas (V_2) **varies directly** with the temperature (T_2).

The behavior of gases, under certain controlled conditions, can also be expressed by the equation $P_1V_1 = P_2V_2$, where P_1 and P_2 denote the pressure of the gas occupying a volume of V_1 and V_2, respectively

Rewriting the equation $P_1V_1 = P_2V_2$ in terms of V_2 leads to.

$$V_2 = \frac{P_1V_1}{P_2}$$

Here, V_2 is said to vary **inversely** with P_2, which means that as P_2 becomes larger, V_2 becomes smaller, and as P_2decreases, V_2 increases Notice that the value of a fraction increases as the denominator decreases

$$\frac{3}{10}, \frac{3}{9}, \frac{3}{8}, \frac{3}{6}, \frac{3}{3}, \frac{3}{1}, \frac{3}{\frac{1}{2}}, \text{ where, } \frac{3}{10} \text{ is less than } \frac{3}{\frac{1}{2}} \left(\frac{3}{\frac{1}{2}} = 6\right)$$

Conversely, the value of a fraction becomes smaller as the denominator becomes larger

$$\frac{3}{\frac{1}{2}}, \frac{3}{1}, \frac{3}{3}, \frac{3}{6}, \frac{3}{8}, \frac{3}{9}, \frac{3}{10}, \text{ where, } \frac{3}{\frac{1}{2}} \text{ is larger than } \frac{3}{10}$$

The formula for the voltage, E, across that part of a circuit with a current, I, passing through a resistance, R, is $E = IR$ Translated in terms of variation, E is said to **vary jointly** as the product of the current and resistance

6-1 VARIATION

The terminology or language of variation as it describes the relationship of the elements of a formula can be summarized in four general statements

1 $C = \pi d$, circumference, C, **varies directly** with the diameter, or C varies as d or C is **proportional to** d

2 $S = \frac{1}{2}at^2$, distance, S, **varies jointly** with a and the square of t

3 $e = 1/r^2$, illumination, e, is **inversely proportional** to the square of the distance, r, or e varies inversely with the square of the distance

4 $CF = mv^2/r$, centrifugal force, CF, **varies jointly** with the mass and the square of the velocity and inversely with the radius This is also an example of **combined variation**

EXAMPLE 6 A

Translate the given equation into the language of variation

$$I = \frac{nE}{R_1 + nR_2}$$

where I is the current, E the voltage, n the number of cells, R_1 the external resistance, and R_2 the internal resistance of the cells

Solution:

The current varies **directly** with the product of the number of cells and the voltage in the circuit, and **inversely** with the sum of the external resistance and the product of the number of cells and the internal resistance of the cells.

EXERCISES 6-1

Translate the following expressions into the language or terminology of variation.

1. Force = mass × acceleration; $F = ma$
2. Energy = mass × (velocity of light)²; $E = mc^2$
3. Centrifugal force $= \dfrac{(\text{mass}) \times (\text{velocity})^2}{\text{radius of path}}$; $CF = \dfrac{mv^2}{r}$
4. Power $= \dfrac{\text{work}}{\text{time}}$; $P = \dfrac{w}{t}$
5. Illumination $= \dfrac{\text{intensity}}{(\text{distance from source})^2}$; $e = \dfrac{i}{r^2}$
6. Joint resistance, $R = \dfrac{R_1 \times R_2}{R_1 + R_2}$
7. Current $= \dfrac{\text{voltage}}{\text{external resistance} + \text{internal resistance}}$; $i = \dfrac{e}{r_e + r_i}$
8. $\dfrac{\text{force}}{\text{weight}} = \dfrac{\text{acceleration}}{\text{pull of gravity}}$
9. $PV = kRT$
10. $\dfrac{\text{object size}}{\text{image size}} = \dfrac{\text{object distance}}{\text{image distance}}$

6-2 PROPERTIES OF PROPORTIONS

A proportion already has been defined as a statement of equality between two ratios. For the purpose of discussing certain properties, the proportion will be written in general form.

$$\frac{a}{b} = \frac{c}{d} \quad (b \neq 0, d \neq 0)$$

The four elements, a, b, c, and d are called the **terms** of the proportion. *The first and fourth terms, a and d, respectively, are called the extremes and the second and third terms, b and c, are called the means.*

$a/b = c/d$ can also be written as $a : b = c : d$ and is read a is to b as c is to d. The expressions $a/b = c/d$ and $a : b = c : d$ are equivalent.

Several properties of proportions are listed below (it is to be understood that, throughout the discussion, all denominators are other than zero):

1 *The product of the means is equal to the product of the extremes*
If $a/b = c/d$, it follows that $ad = bc$

$$\frac{3}{5} = \frac{21}{35}, \text{ then } 3(35) = 5(21) \text{ or } 105 = 105$$

2 *The terms are proportional by inversion*
If $a/b = c/d$, it follows that $b/a = d/c$

$$\frac{3}{5} = \frac{21}{35}, \text{ then } \frac{5}{3} = \frac{35}{21}$$

3 *The terms are proportional by alternation*
If $a/b = c/d$, it follows that $a/c = b/d$

$$\frac{3}{5} = \frac{21}{35}, \text{ then } \frac{3}{21} = \frac{5}{35}$$

4 *If the second and third terms are equal, the second term is called the mean proportional between the first and fourth terms*
If $a/b = c/d$ and $b = c$, it follows that $a/b = b/d$ and b is the mean proportional between a and d

$$\frac{3}{6} = \frac{6}{12}$$

5 These relationships also exist
If $a/b = c/d$, it follows that $(a + b)/b = (c + d)/d$ and $(a - b)/b = (c - d)/d$

$$\frac{3}{5} = \frac{21}{35}, \text{ then } \frac{3+5}{5} = \frac{21+35}{35}, \text{ or } \frac{8}{5} = \frac{56}{35}, \quad \frac{56}{35} = \frac{7 \times 8}{7 \times 5}$$

Also,

$$\frac{5}{3} = \frac{35}{21}, \text{ then } \frac{5-3}{5} = \frac{35-21}{21}, \text{ or } \frac{2}{3} = \frac{14}{21}, \quad \frac{14}{21} = \frac{7 \times 2}{7 \times 3}$$

EXERCISES 6-2

Complete the proportion $a/b = c/d$ by finding the numerical value of the indicated term

1. $\frac{a}{5} = \frac{9}{15}$

2. $\frac{5}{b} = \frac{15}{9}$

3. $\frac{7}{8} = \frac{c}{4}$

4. $\frac{4+2}{2} = \frac{20+d}{d}$

5. $6:9 = 27:b$

6. $9:6 = 27:c$

7. $a:\frac{1}{2} = 13:\frac{1}{2}$

8. $\frac{1}{2}:b = 14:7$

9. $\frac{98-b}{b} = \frac{9}{5}$

10. $\frac{6}{b} = \frac{b}{6}$

11. $\frac{4}{b} = \frac{c}{9}$

12. $\frac{a}{17} = \frac{17}{a}$

13. $\frac{12}{b} = \frac{b}{2}$

14. $\frac{8+a}{16} = \frac{3}{4}$

15. $\frac{16}{a+8} = \frac{4}{3}$

6-3 FORMULAS AND PROPORTIONS

In the field of technology proportions are used quite extensively when properties must be compared. Foremost in this regard are working drawings prepared by technicians, scaled to proportion, such as $\frac{1}{4}$ size, 1 in. = 5 miles, and so on. Proportions are virtually synonymous with chemical compounds. Here, the mathematical balance is just as critical as the ingredients.

By far the Wheatstone Bridge is the most widely used technique for precision measurement. An unknown resistance is balanced in a circuit with three known resistances according to the relationship $R_x/R_1 = R_3/R_2$, where R_1, R_2, and R_3 are known quantities, whereas R_x is to be measured.

Before two quantities with dissimilar units of measurement can be equated, a factor of conversion must be determined: 1 in. = 2.54 cm, or inches/centimeters = 2.54.

Another method of developing a formula, wherein the need for a factor of conversion arises, is based on the concept of variation within a proportion.

The statement that the circumference of a circle varies directly with the diameter can be written as $C \propto d$, where the symbol $\propto$ means **varies with** or is **proportional to.** $C \propto d$ is **not** an **equation** but rather a **statement of proportionality**. In order for this statement to be translated into an equation, a conversion factor or a **proportionality constant**, k, must be introduced. Once k has been determined, the statement $C \propto d$ can be replaced by the equation $C = kd$.

In this illustration, the value of k has been established and referred to as π, where $\pi = 3.1416$. Thus, $C = kd$ can now be expressed as a meaningful and workable formula: $C = \pi d$.

The proportionality constant can be determined experimentally or mathematically as it relates to the physical properties involved. Often, many experiments or calculations are required before a constant or proportionality is discovered or even suggested. The constant of proportionality may vary also with conditions, such as the pull of gravity with respect to altitude.

From time to time a constant has been known to change in value. Refinement comes about as knowledge of subject matter increases and instruments of measurement become more precise. The value of π has witnessed

many interpretations and at best is still an approximation Again, results are only as reliable as the tools of association

The treatment of **proportional variation** basically resolves itself into two types of problems (1) application of an existing proportion or physical relationship, and (2) adopting a procedure whereby the constant of proportionality can be determined

EXAMPLE 6-B

The formula $P_1V_1/T_1 = P_2V_2/T_2$ can be used to find the volume of gas when both temperature and pressure change Find the volume that 500 cm³ of gas at a temperature of 20°C under a pressure of 700 mm will occupy when the temperature rises to 50°C and the pressure increases to 780 mm of mercury

Solution

$$P_1 = 700 \text{ mm}, P_2 = 780 \text{ mm}, T_1 = 20°\text{C} + 273° = 293°\text{K}$$
$$T_2 = 50°\text{C} + 273° = 323°\text{K}, V_1 = 500 \text{ cm}^3$$

Substituting accordingly,

$$\frac{700 \times 500}{293} = \frac{780V_2}{323}$$

Multiplying both sides of the equation by $\frac{323}{780}$ leads to

$$\frac{7 \times 5 \times 10^4}{293} \times \frac{323}{780} = \frac{323}{780} \times \frac{780V_2}{323}$$

$$V_2 = \frac{35 \times 323 \times 10^4}{293 \times 78 \times 10} = 0\,495 \times 10^3 = 495 \text{ cm}^3$$

EXAMPLE 6 C

Find the unknown resistance that provides a balanced bridge when measured in circuit with the following known quantities $R_1 = 2 \times 10^{-2}$ ohms, $R_2 = 3$ ohms, and $R_3 = 1\,8$ ohms

Solution

$R_x/R_1 = R_3/R_2$, substituting respectively

$$\frac{R_x}{2 \times 10^{-2}} = \frac{1\,8}{3},$$

from which

$$R_x = \frac{2 \times 10^{-2} \times 1\,8}{3} = 1\,2 \times 10^{-2} \text{ ohms } (0\,012 \text{ ohms})$$

EXAMPLE 6-D

The velocity ratio of two mating gears is inversely proportional to their diameters Develop a formula for this condition(Fig 6-1)

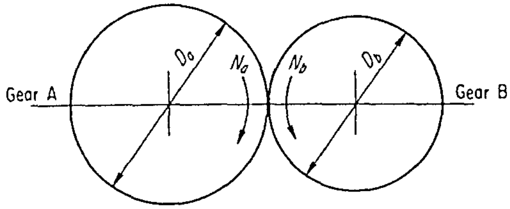


Figure 6-1

D_a, diameter gear A, D_b, diameter gear B
N_a, angular velocity of A (rpm), N_b, angular velocity of B

Solution:

Step 1. Express the conditions of variation in terms of a statement of proportionality. (velocity ratio, $VR = N_a/N_b$)

$$\frac{N_a}{N_b} \propto \frac{D_b}{D_a} \quad \text{or} \quad \frac{N_a}{N_b} = k\frac{D_b}{D_a}$$

Step 2. Solve for the constant of proportionality, k, in terms of given data. The given data may involve dimensional units or mathematical quantities. In this illustration, dimensional units will be used.

$\frac{N_a}{N_b} = k\frac{D_b}{D_a}$; multiplying both sides of the proportion by $\frac{D_a}{D_b}$ yields:

$$\frac{D_a}{D_b} \times \frac{N_a}{N_b} = k\frac{D_b}{D_a} \times \frac{D_a}{D_b}, \quad \text{or} \quad k = \frac{D_a}{D_b} \times \frac{N_a}{N_b}$$

D_a and D_b measured in inches or feet
N_a and N_b measured in revolutions per minute (rpm)
Substituting dimensionally,

$$K = \frac{\text{in.}}{\text{in.}} \times \frac{\text{rpm}}{\text{rpm}}, \text{ which suggests that } k = 1$$

since

$$\frac{\text{in.}}{\text{in.}} = 1 \text{ and } \frac{\text{rpm}}{\text{rpm}} = 1$$

Whenever the units balance out, k will be equal to 1 or some other numerical quantity.

Step 3. Substitute the value of k in the original equation.
$VR = N_a/N_b = 1 \times D_b/D_a$ or $N_a/N_b = D_b/D_a$, which is the formula that gives the relationship between the diameters and angular velocities of two mating gears.

Step 4. Apply the law or developed equation.

EXAMPLE 6-Da:

The velocity ratio of two mating gears (Fig. 6-1) is 1 to 6, or $\frac{1}{6}$. Find the dimension of the small gear if the diameter of the large gear is 9 in.

Solution

$$VR = \frac{D_b}{D_a}, \text{ where } D_a = 9 \text{ in and } VR = \frac{1}{6}$$

Substituting, respectively,

$$\frac{1}{6} = \frac{D_b}{9} \quad \text{or} \quad D_b = \frac{9}{6} = 1\frac{1}{2} \text{ in}$$

EXAMPLE 6-Db

Find N_b when $N_a = 40$ rpm

Solution

$$VR = \frac{N_a}{N_b} \text{ or } \frac{1}{6} = \frac{40}{N_b}, \text{ from which } \frac{N_b}{40} = \frac{6}{1} \text{ (proportion by inversion)}$$

and $N_b = 6 \times 40 = 240$ rpm

EXAMPLE 6-E

Two particles are attracted to each other with a force that is proportional to their masses and inversely proportional to the square of the distance between them The force, F, is calculated in dynes (g-cm/sec^2), mass in grams, and distance, d, in centimeters

Develop a dimensional formula relating to these conditions (known as the gravitational law)

Solution

Step 1 Statement of proportionality

$$F \propto \frac{m_1 m_2}{d^2} \qquad \text{or} \qquad F = k\frac{m_1 m_2}{d^2}$$

Step 2 Solve for k

$$\text{Multiply} \quad F = k\frac{m_1 m_2}{d^2} \quad \text{by} \quad \frac{d^2}{m_1 m_2}$$

$$F \times \frac{d^2}{m_1 m_2} = k\frac{m_1 m_2}{d^2} \times \frac{d^2}{m_1 m_2} \qquad \text{or} \qquad k = \frac{F \times d^2}{m_1 m_2}$$

Substituting dimensional properties,

$$k = \frac{\text{dynes} \times \text{cm}^2}{\text{g} \times \text{g}} = \text{dynes} \times \frac{\text{cm}^2}{\text{g}^2}$$

Step 3 Substitute k in equation of variation

$$F = \text{dynes} \times \frac{\text{cm}^2}{\text{g}^2} \times \frac{m_1 m_2}{d^2}$$

This constant was determined experimentally and is taken as

$$6.67 \times 10^{-8} \text{ dynes } \frac{\text{cm}^2}{\text{g}^2}$$

Checking for dimensional accuracy,

$$F = \text{dynes} \times \frac{\text{cm}^2}{\text{g}^2} \times \frac{\text{g} \times \text{g}}{\text{cm}^2} = \text{dynes} \times \frac{\text{cm}^2}{\text{g}^2} \times \frac{\text{g}^2}{\text{cm}^2} = \text{dynes}$$

EXAMPLE 6-Ea:

Application of law.
Find the gravitational attraction between the earth and the moon.

$$m_1 \text{ (earth)} = 6.0 \times 10^{27} \text{ g}$$
$$m_2 \text{ (moon)} = 7.3 \times 10^{25} \text{ g}$$
$$k = 6.7 \times 10^{-8} \text{ dynes} \times \frac{\text{cm}^2}{\text{gm}^2}$$
$$d = 3.8 \times 10^{10} \text{ cm}$$

Solution:

Substituting accordingly,

$$F = \frac{6.7 \times 10^{-8} \times (6.0 \times 10^{27}) \times (7.3 \times 10^{25})}{(3.8 \times 10^{10})^2}$$
$$= \frac{(6.7 \times 6.0 \times 7.3) \times (10^{-8} \times 10^{27} \times 10^{25})}{14.44 \times 10^{20}}$$
$$= \frac{293.46}{14.44} \times 10^{24} = 20.3 \times 10^{24} \text{ dynes (rounded off)}$$
$$= 2.0 \times 10^{25} \text{ dynes}$$

EXAMPLE 6-F:

The density of a substance is given as 10.5 g/ml. Find the weight, W, of 4 ft³ of this material.

Solution:

One method of solving the problem would be to convert units accordingly. Another is to take advantage of known relationships in application with proportions.

Density = mass/volume and remains contant regardless of quantity. The density of water is taken as 1 g/ml, or 62.4 lb/ft³. Furthermore, the specific gravity of a substance is defined as the ratio of the density of the material to the density of water and is the same for a particular substance, regardless of units of measurement, if there is dimensional consistency.

Thus,

$$\text{specific gravity} = \frac{D_1}{\frac{1 \text{ g}}{\text{ml}}} = \frac{D_2}{\frac{62.4 \text{ lb}}{\text{ft}^3}}$$

where D_1 is the density given in cgs, whereas D_2 is assigned English units

Using W as the unknown weight and substituting in the proportion, defined by the concept of specific gravity, yields this relationship

$$\frac{\dfrac{10\ 5\ \text{g}}{\text{ml}}}{\dfrac{1\ \text{g}}{\text{ml}}} = \frac{\dfrac{W\ \text{lb}}{4\ \text{ft}^3}}{\dfrac{62\ 4\ \text{lb}}{\text{ft}^3}}$$

Clearing dimensional units first,

$$\frac{10\ 5}{1} = \frac{\dfrac{W}{4}}{62\ 4} \quad \text{or} \quad 10\ 5 = \frac{W}{4 \times 62\ 4}$$

Furthermore,

$$W = 10\ 5 \times 249\ 6 = 2{,}620\ 8\ \text{lb} = 2{,}621\ \text{lb}$$

EXAMPLE 6-G

Power is defined as the time rate of doing work Also, power varies inversely with time If the amount of work done in 3 sec is equivalent to 12 hp, how many seconds will it take to develop 15 hp (under identical conditions)?

1 hp is equivalent to 550 ft-lb of work done in 1 sec,

or

$$1\ \text{hp} = 550\ \text{ft-lb/sec}$$

Solution

The approach to this solution will vary slightly from the previous examples

In terms of variation

$$P \propto \frac{1}{t} \quad \text{or} \quad P = \frac{k}{t}$$

The constant of proportionality, k, will remain the same throughout the given discussion These conditions define k as the total work in foot-pounds

Thus, $P = k/t$ or $k = Pt$ holds true for $k = P_1t_1 = P_2t_2$, which also means

$$P_1t_1 = P_2t_2, \text{ where } P_1 = 12\ \text{hp},\ P_2 = 15\ \text{hp},\ t_1 = 3\ \text{sec}$$

and t_2 is the quantity to be computed

From

$$P_1t_1 = P_2t_2,\quad t_2 = \frac{P_1t_1}{P_2}$$

Substituting and solving for t_2,

$$t_2 = \frac{12 \text{ hp} \times 3 \text{ sec}}{15 \text{ hp}} = \frac{12}{5} \text{ sec} = 2\frac{2}{5} \text{ sec}$$

Although the actual value of k was not computed, it did enter into the solution. Regardless of the property of k, the relationship among the conditions of the elements remains unchanged; namely,

$$P_1 = \frac{k}{t_1}, \; P_2 = \frac{k}{t_2}, \; P_1 t_1 = P_2 t_2, \; k = P_1 t_1, \; k = P_2 t_2$$

All of these conditions hold true, no matter how the constant of proportionality is defined.

To illustrate further, the problem will be solved by finding k as in previous examples.

Step 1.

$$P \propto \frac{1}{t} \quad \text{or} \quad P = \frac{k}{t}$$

Step 2.

$$k = Pt$$

$$k = 12 \text{ hp} \times 3 \text{ sec} = 36 \text{ hp-sec.}$$

Step 3.

$$P = \frac{36 \text{ hp-sec}}{t}$$

Step 4.

$$P_2 = \frac{36 \text{ hp-sec}}{t_2}, \quad \text{from which}$$

$$t_2 = \frac{36 \text{ hp-sec}}{P_2} = \frac{36 \text{ hp-sec}}{15 \text{ hp}} = \frac{36}{15} \text{ sec} = 2\frac{6}{15} \text{ sec} = 2\frac{2}{5} \text{ sec}$$

Understanding the principle of a particular concept will usually eliminate unnecessary steps.

EXERCISES 6-3

Express the following relations as:

a. statements of proportionality and,

b. *an equation containing a constant of proportionality (Ex. 1-8).*

1. The coefficient of friction, μ, is inversely proportional to the normal force N.

2. Liquid pressure, p, varies directly with the depth, h, and density, D.

3. The coefficient of fraction, μ, varies with the fractional force, f, and inversely as the normal force, N

4. Illumination, e, varies with the intensity, I, and inversely with the square of the distance, R

5. Electrical power, P, varies jointly with the square of the current, I, and the resistance, R

6. Force, F, varies jointly with the weight, W, and the difference in velocities, V_2, V_1, and inversely with the pull of gravity, g, and time, t

7. The velocity ratio, VR, is inversely proportional to the number of teeth on each gear, N_b, N_a (mating gears)

8. The mechanical advantage of a hydraulic press varies directly with the square of the diameter of the large piston and inversely with the square of the diameter of the small piston

Solve As Indicated

9. The linear velocity, v, of a point on a rotating body varies directly with the distance of the point, r, from the center of rotation and angular velocity, n, of the body
Find the constant of proportionality if

$$v = 176 \text{ in /sec}, \; r = 7 \text{ in }, \; n = 4 \text{ revolutions/sec}$$

10. The circular pitch, P_c (distance between corresponding points on adjacent teeth of a gear) varies directly with the pitch diameter, D, and inversely with the number of teeth, N Find k, if $P_c = 1$ in , $D = 14$ in , and $N = 44$

11. The change in pressure, P, of a flowing liquid varies with $\frac{1}{2}$ of the square of the velocity of the fluid Find the constant of proportionality if $P = 14$ lb/ft^2 and $v = 2$ ft/sec

12. The electric resistance, R, of a wire varies with the length and inversely with the square of the diameter of the wire The proportionality constant ρ, is called the specific resistance Find ρ, for a copper wire if $R = 10\ 4$ ohms $l = 625$ ft, and $d = 25$ mils (one mil $= 0\ 001$ in , in this equation the unit for diameter is given in mils, and the unit length, in feet)

13. 1 cm^3 of copper weighs 8 9 g How much will 1 in ^{3} of copper weigh?

14. The density of gasoline is 0 68 g/ml What volume will 1 kg of gasoline occupy?

15. The weight of 1 ft^3 of water is approximately 62 4 lb What is the capacity, in gallons, of 100 lb of water?

16. The stress developed at any point in a beam varies with the distance from the neutral axis If the stress developed in the fibers, 4 in above the neutral axis, is 4,500 lb/in ^{2}, what will be the stress in the fibers $5\frac{1}{2}$ in above the neutral axis?

17. The ratio of two sides of a triangle is $\frac{1}{2}$. Find the length of the shorter side if the longer side measures $7\frac{1}{2}$ in.

18. If the ratio of two sides of a triangle is 1 and one side measures 19.275 in., what is the length of the other side?

19. The density of hydrochloric acid is 1.20 g/ml.
   a. How much will 2 l of this solution weigh?
   b. What volume will 2 g of this solution occupy?

20. At constant pressure, the volume of a gas varies directly with the absolute temperature (°K = 273 + °C). What volume will 75 ml of hydrogen occupy when the temperature increases from 32°F to 122°F?

21. Gold weighs 0.69 lb/in.^{3}. How much will a cylinder of gold 1 in. in diameter and 1 in. in height weigh? $V = \dfrac{\pi d^2 h}{4}$.

22. The linear expansion of a rod varies jointly with the original length and difference in temperature. Find the coefficient of linear expansion (k) for aluminum if a 3-ft aluminum rod expanded 39×10^{-4} in. when the temperature changed from 0°F to 100°F.

23. Given: erg = dyne-centimeter and dyne = gram-centimeter per square second. Derive a formula for energy in units of ergs that has the physical properties of mass, gravitational constant, and distance. All the components are in units of the metric system and involve grams, seconds, and centimeters (dimensional equation).

24. The intensity of illumination at a certain point varies directly with the intensity of the source of light and inversely with the square of the distance from the source. A lamp with an intensity of 40 candles at 40 cm produces the same illumination as another souce of light at 80 cm. Find the intensity of the second source.

25. A free-falling body covers a distance proportional to the square of the time during which it is moving. If an object falls 256 ft in the first 4 sec, how far will it travel during the next 4 sec?

26. The square of the period of a pendulum (time it takes the pendulum to complete one cycle swing) varies directly with the length of the pendulum and inversely with the pull of gravity.
   a. Find the dimensional units of the proportionality constant in both the metric and English system of measurement. Pull of gravity may be taken as 9.8 m/sec^2 or 32 ft/sec^2.
   b. Find the numerical value of the constant if $t^2 = 121/158$ sec^2, l = 7 in.

UNIT II

Essentials of Algebra

Algebra is a mathematical device that can be used in nearly all branches of mathematics and engineering to reduce complex symbols into meaningful (manageable) relationships. This phase of mathematics will provide those experiences that lead to a fuller understanding of technical concepts. Physical principles cannot be mastered without first acquiring a background involving the essentials of algebra.

This unit includes the study of various definitions, laws, and techniques associated with algebra. Included are the operations involving signed numbers and all those procedures that lead to the solution of linear equations and quadratics equations, as well as the plotting of various functions.

CHAPTER 7

Preliminary Concepts

The number system with which the technician will be working is considered an orderly system. To pursue this notion further, the earlier concept (Secs. 1-3 and 1-4) of representing numbers by corresponding points on a line will be reproduced in Fig. 7-1.

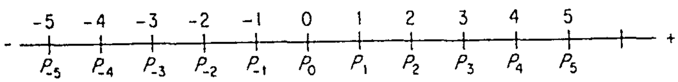


Figure 7-1

Starting with any point, or number, and proceeding to the right (positive direction), each successive number is larger than the preceding one.

$8 > 5, 5 > 4, -2 > -3, \ldots,$ where the symbol $>$ means *larger than*; i.e., 8 is larger than 5.

Likewise, going to the left (negative direction), each number is smaller or less than the preceding one.

$5 < 8, 4 < 5, -3 < -2, \ldots,$ where the symbol $<$ means *less than*; i.e., 5 is less than 8.

Furthermore, the spaces or units between consecutive points are equal. The distance between P_2 and P_3 is equal to the distance between P_4 and P_5. Basically, this is what is meant by an **orderly system**. The numbers, or points, are **not arranged haphazardly**, but rather follow a pattern of **consistency** with respect to numerical value (quantity) and direction.

Numbers were originally devised to facilitate counting, to keep account of possesions via the concept of quantity. Later the concept was extended to define tracts of land, thus introducing the concept of measurement.

Quantitatively, possessions did not keep accumulating forever. There were various losses, which now come to be recognized as deficits and are recorded in some manner as negative. The same is true in the case of measure-

ments Several reversals in direction are required to completely define a plot of land These changes in direction are also recorded as positive and negative Thus, the introduction of signed numbers

7-1 SIGNED NUMBERS

Signed numbers are numbers that are identified as either positive or negative $+2, -7, +10, +\pi, -3\,250, +\frac{3}{4}, -8\frac{2}{3},$ It is customary to omit the (+) sign for numbers assumed positive, unless their identity, for some reason, has to be emphasized This is not the case for negative numbers however Furthermore, negative numbers are normally enclosed within parentheses (-9) $(-13\,7)$, thus assuring proper identification and distinguishing this notation from the arithmetic operation of subtraction

This leads presently to the discussion of applying the fundamental arithmetic operations of addition, subtraction multiplication, and division to signed numbers

7-2 ADDING SIGNED NUMBERS

The symbols (+) and (−) now take on double meaning First of all, they are associated with the addition and subtraction of quantities Second they have just been assigned the concept of direction with respect to the identity of a number Thus the distinction between operation and direction must always remain clear For example,

$$34 + (-12), (-17) - (+79), (-129) + (-43)$$

The addition of signed numbers can be demonstrated with the use of the number scale (Fig 7-2)

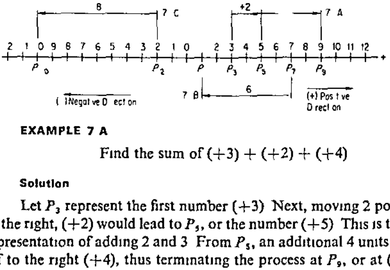


Figure 7-2

EXAMPLE 7 A

Find the sum of $(+3) + (+2) + (+4)$

Solution

Let P_3 represent the first number $(+3)$ Next, moving 2 points or units to the right, $(+2)$ would lead to P_5, or the number $(+5)$ This is the graphical representation of adding 2 and 3 From P_5, an additional 4 units are counted off to the right $(+4)$, thus terminating the process at P_9, or at $(+9)$

$$\text{Hence, } (+3) + (+2) + (+4) = +9, \text{ or } 9$$

The process just demonstrated is referred to as **graphic addition** or **geometric addition.**

EXAMPLE 7-B:

Find the sum of $(+7) + (-6)$.

Solution:

The starting point this time will be P_7, or $(+7)$, Fig. 7-2. To add (-6) to $(+7)$ graphically means to move 6 units (-6) to the left of P_7. The terminal point of this addition will be P_1, which corresponds to the number $(+1)$.

Thus it appears that $(+7) + (-6) = 1$

EXAMPLE 7-C:

With the use of Fig. 7-2, add (-2) and (-8).

Solution:

Here, the operation starts at P_{-2}, identifying the number (-2). Adding (-8) to (-2) graphically indicates counting off 8 units to the left (-8) of P_{-2}. The operation terminates at P_{-10}, or at (-10). Therefore,

$$(-2) + (-8) = -10$$

The concept of signed numbers leads to another principle, that of absolute value. From Fig. 7-2, it is apparent that the distance from the origin, O, to P_{12} $(+12)$ is equal to the distance from O to P_{-12} (-12). That is to say, the distances are equal in magnitude (measurement) although opposite in direction. Mathematically, however, $+12$ is not equal to -12 $(+12 \neq -12)$. Thus, to provide for this condition, the symbol $|\quad|$ is used, which is associated with the term, **absolute value.**

Absolute value refers to the quantitative value of a number without reference to sign. In terms of symbols,

$$|+12| = |-12| = 12$$

It must be emphasized that the expression

$$|+12| = |-12| = 12$$

does not indicate that $+12 = -12$; it simply points out that in terms of magnitude, $+12$ and -12 are identical.

The order in which numbers are added will not affect the sum: $4 + 3 = 3 + 4$, or $(+5) + (+12) + (-7) = (-7) + (+12) + (+5)$. This is known as the *commutative law* of addition. In general form, the law is written accordingly:

$a + b = b + a$, where a and b are arbitrary numbers.

The graphical method of adding and subtracting signed numbers was introduced only to demonstrate a principle. The algebraic method is much more expedient, especially when the numbers are other than integers.

Rule: Adding of Signed Numbers:

1 *To add numbers of like sign, add their absolute values and prefix the sum with the common sign*

$$(+7) + (+3) + (+6) = |+7| + |+3| + |+6|$$
$$= 7 + 3 + 6 = 16$$
$$(-7) + (-3) + (-6) = |-7| + |-3| + |-6|$$
$$= 7 + 3 + 6 = 16$$

But the common sign is minus Therefore, $(-7) + (-3) + (-6) = -16$

2 *To add numbers of unlike signs, convert numbers to absolute values then subtract the smaller from the larger Prefix the sum with the sign of the number that is larger in absolute value*

It must be pointed out that reference here to smaller and larger is made in the context of absolute values $(-7) < (+4)$, however $|-7| > |+4|$

EXAMPLE 7-D:

Add (-7) and $(+4)$

Solution:

Following the principle of rule 2,

$$(-7) + (+4) = |-7| - |+4| = 7 - 4 = 3$$

But the sign of the number that is larger in absolute value is negative (-7) Therefore,

$$(-7) + (+4) = -3$$

EXAMPLE 7-E·

Find the sum of $(-13) + (+8) + (-5) + (+3)$

Solution

Combine terms according to signs, that is, add all negative terms and all positive terms first (Commutative Law)

$$(-13) + (+8) + (-5) + (+3) = [(-13) + (-5)] + [(+8) + (+3)]$$

where

$$[(-13) + (-5)] = -18 \quad \text{and} \quad [(+8) + (+3)] = +11$$

These terms are then added, leading to completion of the problem

$$(-18) + (+11) = |18| - |11| = 7$$

Since $|18| > |11|$, the negative sign will prefix the sum. Thus,

$$(-18) + (+11) = -7$$

or

$$(-13) + (+8) + (-5) + (+3) = -7$$

The problem could also be solved by taking the terms consecutively, combining each term with the cumulative sum of the preceding terms.

$$\begin{aligned}&(-13) + (+8) + (-5) + (+3)\\ &= (-5) \quad + (-5) + (+3)\\ &= (-10) \quad + (+3)\\ &= -7\end{aligned}$$

7-3 SUBTRACTING SIGNED NUMBERS

Basically, subtraction is an arithmetic operation that involves finding the difference between two numbers or quantities, such as the difference $(+32) - (+145) = d$. Here, 32 is called the *minuend*, the number 145 is termed the *subtrahend*, and d is the *difference.*

Another way of defining subtraction is to view it as a process of finding a number, d, **(difference)** such that when it is **added** to the **subtrahend**, the **sum**, will be **equal** to the **minuend**. In terms of the specific problem, the preceding statement can be written mathematically as:

$$d + 145 = 32, \text{ where } d \text{ is equal to } (-113).$$

Thus,

$$(+32) - (+145) = -113$$

Based on the definition of arithmetic subtraction and the principle of signed numbers, the following rule applies to the subtraction of signed numbers:

Rule: Subtracting Signed Numbers. *To subtract quantities involving signed numbers, change the sign of the subtrahend (either + to —, or — to +) and add according to the rules adopted for adding signed numbers.*

Applying this rule to the preceding problem leads to this procedure;

Subtraction: $(+32) - (+145)$

Rule: *Change the sign of the subtrahend from* $+145$ *to* -145 *and add.*

Addition: $(+32) + (-145) = -113$

In summary, the subtraction of signed numbers is really a two-step process: (1) change the sign of the subtrahend and, (2) with the change in sign, add in accordance with the established rules for adding signed numbers.

Subtraction $(+32) - (+145)$ leads to
Addition $(+32) + (-145) = -113$

EXAMPLE 7-F.

Given $(+76) - (+25)$, find the difference

Solution

Change the sign of the subtrahend from $+25$ to -25 and add Thus,

$$(+76) - (+25) = (+76) + (-25) = +51$$

EXERCISES 7-1

Perform the indicated operations (Ex 1 20)

1. $(+4) + (-4) =$
2. $(-4) + (+4) =$
3. $(-4) + (-4) =$
4. $(+4) - (-4) =$
5. $(+10) + (-9) + (-3) =$
6. $(-7) - (+7) + (+8) + (-8) =$
7. $(+1) - (-5) + (-1) =$
8. $(+16) - (-18) + (-18) =$
9. $(-20) - (-20) - (+20) - (-20) =$
10. $(+793) - (+937) =$
11. $(+2) - (-8) + (-8) + (-2) + (-3) =$
12. $(+73) + (-86) - (-44) + (-51) - (+32) =$
13. $(+25) - (-25) + (-50) =$
14. $(-15) + (+30) - (-20) + (-5) =$
15. $(+76) - (n) = -10$ Find n
16. $(m) - (27) = 14$ Find m
17. $(-17) - (p) = 0$ Find p
18. $(-12) - (+15) + (r) = -2$ Find r
19. $(s) - (-7) - (+7) + (-14) = 0$ Find s
20. $(+10) - (-8) + (-11) - (+7) + (t) = 0$ Find t
21. At 9 00 AM the temperature reading was $+42$ °F By 1 00 PM the temperature dropped 57°F What was the reading at 1 00 PM?
22. If the temperature is -21°C and later rises to 37°C, what is the total rise?
23. The temperature range for a 24-hour period was 29°F If the high for the day was 13°F, what was the corresponding low?
24. The temperature range during a 12-hour period was 17°C The low reading was -5°C Find the high

Signed numbers are multiplied and divided, basically, according to the procedure established for the corresponding arithmetic operations. The added element of signs (+ and −), however, brings in another consideration.

Multiplication involves quantities called **factors** that are **combined** in a process that results in a **product.**

$(+5) \times (+7) = 35$, which can be interpreted as adding $+7$ five times.

$$(+5) \times (+7) = (+7) + (+7) + (+7) + (+7) + (+7) = +35$$

Applying the same principle to the multiplication of $(+5) \times (-7)$ leads to the following product:

$$(+5) \times (-7) = +(-7) + (-7) + (-7) + (-7) + (-7) = -35$$

If in the immediate multiplications $[(+5) \times (+7)$ and $(+5) \times (-7)]$ the interpretation of the factor $(+5)$ leads to adding a second factor 5 times, then it follows that in the multiplication $(-5) \times (-7)$, (-5) should be associated with the process of subtracting a second factor 5 times to obtain the product. Hence,

$$(-5) \times (-7) = -(-7) - (-7) - (-7) - (-7) - (-7) = +35$$

or,

$$(-5) \times (+7) = -(+7) - (+7) - (+7) - (+7) - (+7) = -35$$

These illustrations were intended to serve as an introduction to the rules for multiplying signed numbers.

Rule: Multiplication of Signed Numbers

1. *If two factors are multiplied together, both of which have like signs (both + or both −), the product will be positive.*

2. *If two factors of unlike signs are multiplied together, their product will be negative.*

EXAMPLE 7-G:

Multiply the respective quantities.

a. $(+12) \times (+5)$ b. $(+12) \times (-5)$

c. $(-12) \times (+5)$ d. $(-12) \times (-5)$

Solution:

Since the factors in Ex. 7-Ga and Ex. 7-Gd have like signs, respectively, their product will be positive.

a. $(+12) \times (+5) = +60$ and d. $(-12) \times (-5) = +60$

The factors of the remaing two problems have unlike signs; thus,

b. $(+12) \times (-5) = -60$ and c. $(-12) \times (+5) = -60$

From Ex 7-G, if $(+12) \times (+5) = 60$, it follows (based on the arithmetic concept that division is the inverse of multiplication) that

$$\frac{+60}{+5} = +12$$

Also,

if $(+12) \times (-5) = -60$, it follows that $\frac{-60}{-5} = +12$, and

if $(-12) \times (+5) = -60$, it follows that $\frac{-60}{+5} = -12$, finally,

if $(-12) \times (-5) = +60$, then $\frac{+60}{-5} = -12$

This generalization leads to the rule for dividing signed numbers

Rule. Dividing Signed Numbers

1 *If two quantities are divided, both of which have like signs, their quotient will be positive* (divisor $\neq 0$)

2 *If two quantities of unlike signs are divided, their quotient will be negative* (divisor $\neq 0$)

EXERCISES 7-2

Perform the indicated operations (Ex 1-20)

1. $(+4) \times (+10) =$ *2.* $(-5) + (+17) =$

3. $(-30) - (+6) =$ *4.* $(-3) \times (-5) \times (-4) =$

5. $(+34) - (-17) =$ *6.* $(-63) - (-9) =$

7. $(-1) \times (-2) \times (-3) \times (-4) =$

8. $(+2) \times (-8) \times (-2) =$

9. $(+144) - (-6) =$

10. $(+3) \times (-3) \times (+3) \times (-3) =$

11. $(+5) \times (-6) \times (n) = +90$ Find n

12. $(m) - (-14) = +9$ Find m

13. $(-124) - (p) = +31$ Find p

14. $\frac{(-3) \times (-7)}{(-21)} =$ *15.* $\frac{(+4) \times (+13)}{(-8)} =$

16. $\frac{(-2) \times (r)}{(-12)} = (-42)$ Find r

17. $\frac{(-13) \times (+5)}{(s)} = (-5)$ Find s

18. $(-2) \times (-4) + (-3) \times (-8) =$

19. $\left(-\frac{3}{4}\right) \times \left(-\frac{16}{9}\right) =$

20. $\left(-\frac{5}{12}\right) \times \left(+\frac{60}{7}\right) =$

7-5 SYMBOLS OF GROUPING

The expression $2 \times 3 + 4$ might be combined by multiplying the first two terms and then adding the last term to this product:

$$2 \times 3 + 4 = 6 + 4 = 10;$$

or, the last two terms could be added and their sum multiplied by the first term:

$$2 \times 3 + 4 = 2 \times 7 = 14$$

Apparently one of these solutions is incorrect, which suggests that expressions involving multi-arithmetic operations require some form of guidelines such that the problem can be approached with consistency. The following rule attempts to do just that.

Rule: *To reduce or simplify an expression containing several combined and related arithmetic operations,*

1. *Complete the indicated multiplication and/or division first, followed by*
2. *Addition and/or subtraction, in the order of appearance, in the expression.*

Accordingly, $2 \times 3 + 4 = 6 + 4 = 10$

Perhaps a more effective way of clarifying the intent of a mathematical expression is through the use of **symbols of grouping. Symbols of grouping** are used to indicate or emphasize that certain **terms** or **factors** of a mathematical expression are to be considered as a **single quantity**. The usual symbols of grouping are; **(parentheses), [brackets], {braces}**, and the $\overline{\textbf{vinculum}}$, along with the $\sqrt{\textbf{radical}}$.

With reference to the example $2 \times 3 + 4$, use of parentheses clarifies the distinction between

$$(2 \times 3) + 4 = 10$$

and

$$2 \times (3 + 4) = 2 \times 7 = 14$$

An expression containing symbols of grouping can be simplified by performing the indicated operation(s) within the parentheses or other form of grouping, and then combining this quantity with the other terms, according to the following rules:

A plus sign preceding an expression enclosed by parentheses or other form of grouping indicates that every term within the parentheses is to be multiplied by $+1$.

$$+(2 - 7) = +2 - 7$$

A minus sign preceding an expression enclosed by parentheses means that every term within the parentheses, or other symbol of grouping, is to be multiplied by -1

$$-(2-7) = -2+7$$

In general form,

$$+a(b+c) = +ab+ac$$

and,

$$-a(b+c) = -ab-ac$$

where, a, b, and c are real numbers The above expansions demonstrate the *distributive law for multiplication* This is also referred to as *removing parentheses*

The technician will be confronted with many engineering relationships that involve several terms arranged according to various concepts called formulas or laws

$$\omega = \sqrt{\frac{2g}{r}(1-\cos \text{O})}, \quad Z = \sqrt{R^2 + (X_L - X_C)^2}$$

The first equation involves angular velocity and the second defines the impedance of an electric circuit They are included here as an example of symbols of grouping

In simplifying or reducing expressions involving several grouping symbols, it is common practice to start with the innermost set and continue outward until all symbols are removed The terms are then combined according to the indicated mathematical operation If the parentheses are not separated by a sign, it is understood that the quantities are to be multiplied [$(-3)(2) = -6$]

EXAMPLE 7-H

Simplify the following expression $-3[2(6-7)-4(-5+7)]$

Solution

First, reduce terms associated with parentheses

$$\begin{aligned} -3[2(6-7)-4(-5+7)] &= -3[2(-1)-4(+2)] \\ &= -3[-2-8] \end{aligned}$$

Thus, the parentheses have been removed Next, combine the remaining terms within the brackets and multiply by -3

$$-3[-2-8] = -3[-10] = +30$$

Hence,

$$-3[2(6-7)-4(-5+7)] = +30$$

Notice that the following expressions are equivalent:

$$-3[-2-8] = -3(-2-8) = -3\{-2-8\}$$

Alternate Solution:

Remove parentheses by multiplying through by the respective factors.

$$-3[2(6-7) - 4(-5+7)] = -3[+12 - 14 + 20 - 28]$$

Combine terms within brackets.

$$-3[+32 - 42] = -3[-10] = +30$$

EXAMPLE 7-1:

Simplify · $5\{-2[3 + 6(-2+5)] - 4[7 - (4-1) + 3]\}$

Solution:

Remove parentheses, brackets, and braces in that order.

$$5\{-2[3 + 6(-2+5)] - 4[7 - (4-1) + 3]\}$$
$$= 5\{-2[3 + 18] - 4[7 - 3 + 3]\} = 5\{-2[21] - 4[7]\}$$
$$= 5\{-42 - 28\} = 5\{-70\} = -350$$

It is unlikely that the technician will be involved with formulas or other mathematical expressions that will contain more than two symbols of grouping in any immediate situation. Most of the engineering relationships will involve only the radical, $\sqrt{}$, and parentheses ( ).

EXERCISES 7-3

Simplify the following expressions. (Remove the symbols of grouping.)

1. $-(3+5) + (-3-5)$
2. $-2[-(3+5) + (-3-5)]$
3. $(6+2) - (6-2) + (-6-2)$
4. $-5[(6+2) - (6-2)] + 5(6-2)$
5. $+3(7-5) + 3\{-3(7-5) + 3[(10+2) - 5] + 3(7-5)\}$
6. $-2(16-12) + 2(16-12) - 3(9+2) - 4[2 - (7-12) + (5-1)]$
7. $-2\{4 + 3(6-4) + 4(1-9) - [4 + 3(6-4) + 4(1-9)] + 10\}$
8. $3(-10+13) - 7(-4+5) + 8[-1 - (4-6) - 2(3-2) + 1]$
9. $-7\{4 - [(-9+2) + (2-9) + 4] + 2 - (-9+2)\} + (2-9)$
10. $-\{(-12+4) + 2[-(-13+9) - 6(7-3)] + 2(17-5)\} - 2$

7-6 ALGEBRAIC EXPRESSIONS

An expression such as $°F = \frac{9}{5}°C + 32°$ is a *formula* that is used in science and engineering to convert, (°C), Centigrade temperature readings to

(°F) Fahrenheit readings On the other hand, the statement, $y = \frac{9}{5} \times + 32$ is called an *algebraic equation* Both expressions give the same mathematical relationship with respect to two variables In the first expression, the variables are °C and °F, whereas the second introduces the conventional, x and y

Whenever a formula is developed, an attempt is made to use the first letter of the concept as a representative symbol

$C = 2\pi r$, where C is the circumference and r is the radius

$v = gt$ where v is the velocity, g is the pull of gravity and t is the time

In the field of mathematics, the letters belonging to the first part of the alphabet, a b, c, d, , are used to represent *constants*, whereas the letters at the end of the alphabet are used to denote *variables*, x, y, and z

In the formula $C = 2\pi r$, the circumference varies with the radius Notice, however, that 2π remains unchanged regardless of the dimensions of the circle Thus, r and C are variables, that is, they can be assigned different values to meet various conditions

$C = 2\pi r$, let $r = 1\ 0$ in Then

$C = 2\pi(1\ 0) = 2(3\ 14)(1\ 0) = 6\ 28$ in Next, let $r = 3\ 0$ in

Then it follows that

$C = 2(3\ 14)(3\ 0) = 6\ 28(3\ 0) = 18\ 84$ in

Note 2π remained constant and equal to 6 28

A *constant* is a term whose numerical value remains the same regardless of other conditions The temperature may be 20°C at one time of the day and 75°F an hour later, but 32° in the formula $°F = \frac{9}{5}°C + 32°$ remains constant Likewise, the factor $\frac{9}{5}$ is considered a constant

A *variable* is a term that can assume, by design or other conditions, different values ($r = 1\ 0$ in , $r = 3\ 0$ in , etc)

The following symbols, which include constants and variables, are referred to as *algebraic expressions*

$2x, 4y, 5x - 1, ax + by, 2x^2 + 3xy + 4y^2$, etc

The numerical portion of an algebraic term is called the coefficient In the term $3xy$, 3 is called the coefficient of xy

The statements below are algebraic expressions that are called **algebraic equations**

$2x = 3, x + 5y = 7, 3y = -4, x^2 - y^2 = 25, ax + by - c = 0$, etc

The technician will be most concerned with formulas or equations, although other algebraic expressions will be used on occasion to define some relevant concept

Algebraic expressions are classified according to the number of terms involved and the degree of the variable

Monomials: one term, $2x$, $7xy^2$, $\sqrt{3}\,xy$, $\frac{5}{8}z$, ...

Binomials: two terms, $2x - y$, $x^2 + y^2$, $ax + by$, ...

Trinomials: three terms, $2x - y + z$, $x^2 - y^2 + z^2$, ...

The sum of one or more monomials is called a polynomial.

$x^2 + y^2 - 7xy + 36$, $2xy + 4xz + 12z^2 - 15x - 15y$, ...

Polynomials are expressions that include **binomials** and **trinomials**.

7-7 ADDITION AND SUBTRACTION OF POLYNOMIALS (ALGEBRAIC EXPRESSIONS)

Algebraic expressions can be added or subtracted much the same as arithmetic expressions. Recall that arithmetic expressions are combined (by addition or subtraction) only if the units are identical: 33.0 ft + 77.2 ft, 2.1×10^{-3} g $- 1.75 \times 10^2$ g, 11.2 cm + 3.7 cm, etc.

Applying this principle to algebraic expressions:

$$5x + 2x - 4x = 3x$$

$$7y + 5y - 12y = 0, \text{ etc.}$$

Basically, algebraic terms are made up of a number and one or more letters: $2x$, $-3ab$, $\sqrt{5}\,xyz$, πd, ... These letters are called *literal numbers* and usually represent numbers.

In order to combine algebraic expressions by addition or subtraction, the literal portion of the terms must be identical: $2x$ and $7x$, $351z$ and $9z$, $-2xy$ and $13xy$, and so on. Identical terms are then treated or combined in the same manner as arithmetic quantities. Expressions that are not identical are left as an indicated sum or difference.

$$3x + 7x - 5y - 2y + z = 10x - 7y + z$$

A coefficient of 1 is seldom used: $x = 1x$, $1y = y$, ...

EXAMPLE 7-J:

Find the sum of the given expressions (polynomials).

$$(2xy - 3y + 7) \quad \text{and} \quad (4xy + 5y - 6)$$

Solution:

Place one expression over the other so that the columns are made up of identical terms; then procede with the algebraic addition.

$$\begin{array}{r} 2xy - 3y + 7 \\ \underline{4xy + 5y - 6} \\ 6xy + 2y + 1 \end{array}$$

which is the sum of the given polynomials.

Alternate Solution

Remove parentheses and combine like terms

$$(2xy - 3y + 7) + (4xy + 5y - 6) = 2xy + 4xy - 3y + 5y + 7 - 6$$
$$= 6xy + 2y + 1$$

EXAMPLE 7-K

Find the difference $(4xy + 5y - 6) - (2xy - 3y + 7)$

Solution

Place one expression over the other such that the columns contain corresponding terms

Subtract:

$$\begin{array}{r} 4xy + 5y - 6 \\ \underline{2xy - 3y + 7} \end{array}$$

Change sign of subtrahend and add

Add:

$$\begin{array}{r} 4xy + 5y - 6 \\ \underline{-2xy + 3y - 7} \\ 2xy + 8y - 13 \end{array},$$

which is the difference of the given polynomials

Alternate Solution

Remove parentheses and combine like terms Recall that if parentheses are preceeded by a plus sign (+), the parentheses can be removed without changing the sign of the terms within If parentheses are preceeded by a minus sign (−), they can be removed by changing the signs of the terms within Thus,

$$(4xy + 5y - 6) - (2xy - 3y + 7)$$
$$= 4xy + 5y - 6 - 2xy + 3y - 7 = 2xy + 8y - 13$$

EXAMPLE 7-L

Simplify the given expression by removing symbols of grouping and combining like terms

$$-(ab + 3x - c) - \{c + 4 + [6ab - (6ab - x)] + 7\}$$

Solution

$$-(ab + 3x - c) - \{c + 4 + [6ab - (6ab - x)] + 7\}$$
$$= -ab - 3x + c - \{c + 11 + [x]\} = -ab - 3x + c - \{c + 11 + x\}$$
$$= -ab - 3x + c - c - 11 - x = -ab - 4x - 11,$$

which is the simplified form of the given expression. Another way of writing the reduced form is to factor out a minus sign. Hence,

$$-ab - 4x - 11 = -(ab + 4x + 11)$$

EXERCISES 7-4

Add the following polynomials (Ex. 1-6).

1. $\begin{array}{l} 2a - 7b + 3 \\ \underline{4a + 3b - 2} \end{array}$

2. $\begin{array}{l} -2a + 7b - 3 \\ \underline{-4a - 3b + 2} \end{array}$

3. $\begin{array}{r} 5xy + 3x - 4y \\ \underline{-2xy - 3x + 4y} \end{array}$

4. $\begin{array}{r} -5xy - 3x + 4y \\ \underline{2xy + 3x - 4y} \end{array}$

5. $\begin{array}{l} \quad 7a + 3b - 4c + 5 \\ \underline{-4a + 7b + 5 \qquad\quad} \end{array}$

6. $\begin{array}{l} 14b - 7c + 13 \\ \underline{15a - 12b + 3c - 13} \end{array}$

Subtract the bottom expression from the polynomial that appears on top (Ex. 7-12).

7. $\begin{array}{r} 12x - 13y + 7 \\ \underline{5x + 17y + 12} \end{array}$

8. $\begin{array}{r} -12x + 13y - 7 \\ \underline{-\ 5x - 17y - 12} \end{array}$

9. $\begin{array}{r} 21a - 15b + \ 6c \\ \underline{-32a + \ 9b - 10c} \end{array}$

10. $\begin{array}{r} -32a + \ 9b - 10c \\ \underline{21a - 15b + \ 6c} \end{array}$

11. $\begin{array}{r} 45xy - 17y + 32x - 5 \\ \underline{+ 19y - 16x + 5} \end{array}$

12. $\begin{array}{l} 7x + 14y - \ 5z + 10 \\ \underline{\quad + 15y + 16z \qquad} \end{array}$

Simplify the following algebraic expressions by removing the symbols of grouping and combining like terms.

13. $a + (2 - 5a)$

14. $a - (2 - 5a)$

15. $3b + (6a - 3b) + 4a$

16. $3b - (6a - 3b) + 4a$

17. $-(a + 2b) + (a - 2b)$

18. $-(3 + 4x + 7y) - (7x + 4y)$

19. $-[3x + (y - 2x) + 5] + 6 - [2x - (3y + x) + 5]$

20. $2x - \{-[-(3x + 2y - 6) + (2x - 3y + 6)] + (5y - 5)\} - 2x$

21. $(2y - x) - [-(3x + 2) + (2y - 5)] + [2 + (3x + 4y) - (3y + 7)]$
$+ (2x - y)$

22. $(2x + 3) - \{-3 + 3[(4 - y) + 2x] - [5 - (3y - 1)]\}$

7-8 MULTIPLYING AND DIVIDING POLYNOMIALS

Multiplying algebraic expressions involves procedures adopted for multiplying arithmetic quantities and includes, further, the Law of Signs, the Laws of Exponents, and the Distributive Law for Multiplication.

$$2 \times 3a = 6a; \ -3(x - 5y) = -3x + 15y$$

If the literal parts of the factors are unlike, they are combined as an indicated product

$$a \times b = ab, \ (x)(y)(z) = xyz$$

If the literal portions are identical, they are combined according to the laws of exponents

$$(3a)(4a) = (3)(4)(a \times a) = 12a^2$$

The **Laws of Exponents** are summarized below (see also Sec 4-2)

$$(a^m)(a^n) = a^{m+n}, \text{ for } m = 3 \text{ and } n = 5$$
$$(a^3)(a^5) = a^8 \text{ or } (aaa)(aaaaa) = a^8$$
$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}}, \text{ if } m = 3 \text{ and } n = 5$$
$$\frac{a^3}{a^5} = a^{3-5} = a^{-2} = \frac{1}{a^{5-3}} = \frac{1}{a^2}$$

where

$$a^{-2} = \frac{1}{a^2}$$
$$a^0 = 1 \text{ if } a \neq 0$$

Not included previously are the forms $(a^m)^n$, *which is called the power of a power*, $(ab)^n$, *the power of a product*, and $(a/b)^n$, *the power of a quotient*

$$(a^m)^n = a^{mn}, \text{ for } m = 3 \text{ and } n = 5$$
$$(a^m)^n = (a^3)^5 = (a^3)(a^3)(a^3)(a^3)(a^3) = a^{15}$$

or

$$(a^3)^5 = a^{3\times 5} = a^{15}$$
$$(ab)^n = a^n b^n, \text{ for } n = 3$$
$$(ab)^n = (ab)^3 = (ab)(ab)(ab) = a^3 b^3$$
$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \text{ where } b \neq 0$$

Along with the Distributive Law and the Laws of Exponents, the *Commutative Law for Multiplication* is also applicable

$$ab = ba$$

EXAMPLE 7-M

Perform the indicated multiplication

$$3x^2y(2x - 4xy - 5xy^2 + x^2y^2)$$

Solution:

Multiply each term of the expression $(2x - 4xy - 5xy^2 + x^2y^2)$ by $3x^2y$

$$(3x^2y)(2x) - (3x^2y)(4xy) - (3x^2y)(5xy^2) + (3x^2y)(x^2y^2)$$
$$= 6x^3y - 12x^3y^2 - 15x^3y^3 + 3x^4y^3.$$

Since none of the terms are identical, the product cannot be simplified further.

Summarizing, to multiply one polynomial (or monomial) by another, multiply each term of the first polynomial by each term of the second according to the principles discussed.

EXAMPLE 7-N:

Multiply $(3x - 4)$ by $(2x + 3)$.

Solution:

The procedure for multiplying two factors is fundamentally the same as arithmetic multiplication. In the product 36×24, every digit in the multiplicand, 36, is multiplied by every digit of the multiplier, 24. Similarly, every term of the factor $(3x - 4)$ must be multiplied by every term of $(2x + 3)$.

$$\begin{array}{r}
36 \\
\underline{24} \\
144 \\
\underline{72} \\
864
\end{array}
\qquad\qquad
\begin{array}{rrrl}
 & 3x & -\ 4 & \\
 & 2x & +\ 3 & \\
\hline
 & +\ 9x & -\ 12 & \longleftarrow 3(3x - 4) \\
6x^2 & -\ 8x & & \longleftarrow 2x(3x - 4) \\
\hline
6x^2 & +\ x & -\ 12 & \longleftarrow \text{(adding respective columns)}
\end{array}$$

During intermediate steps, the products should be placed in columns containing similar terms, such as the column for x^2, x, ...

Alternate Solution:

The product could have been found by applying the distributive principle for multiplication.

$$(3x - 4)(2x + 3) = 3x(2x + 3) - 4(2x + 3)$$
$$= 6x^2 + 9x - 8x - 12 = 6x^2 + x - 12$$

EXAMPLE 7-O:

Find the product of $(x^2 - 3x + 2)(3x^2 + 4x - 2)$.

Solution:

$$\begin{array}{rrrrrl}
 & & x^2 & -\ 3x & +\ 2 & \\
 & & 3x^2 & +\ 4x & -\ 2 & \\
\hline
 & & -\ 2x^2 & +\ 6x & -\ 4 & \longleftarrow (-2)(x^2 - 3x + 2) \\
 & 4x^3 & -\ 12x^2 & +\ 8x & & \longleftarrow (4x)(x^2 - 3x + 2) \\
3x^4 & -\ 9x^3 & +\ 6x^2 & & & \longleftarrow (3x^2)(x^2 - 3x + 2) \\
\hline
3x^4 & -\ 5x^3 & -\ 8x^2 & +\ 14x & -\ 4 & \longleftarrow \text{(adding columns)}
\end{array}$$

Thus, $(x^2 - 3x + 2)(3x^2 + 4x - 2) = 3x^4 - 5x^3 - 8x^2 + 14x - 4$

Division is considered an arithmetic operation, the inverse of multiplication

$$(3a)(6) = 18a, \text{ then } \frac{18a}{3a} = 6$$

In dividing algebraic expressions, the coefficients are combined in the manner of long or short division whereas the literal factors are combined in accordance with the laws of exponents

$$\frac{15a^3}{5a} = \frac{15}{5}\left(\frac{a^3}{a}\right) = 3a^{3-1} = 3a^2$$

EXAMPLE 7 P

Divide $(6x^2 - 5x - 6)$ by $(2x - 3)$

Solution

The process about to be employed is similar to that of dividing 357 by 17 wherein the terms of the polynomials are viewed as being analogous to the digits of the arithmetic expressions

$$\begin{array}{rl} & 21 \\ 17 & \overline{)\,357} \\ & \underline{34} \\ & 17 \\ & \underline{17} \\ & 0 \text{ Remainder} \end{array} \qquad \begin{array}{lrl} & 3x + 2 & \\ 2x - 3 & \overline{)\,6x^2 - 5x - 6} & \\ & \underline{6x^2 - 9x} & \longleftarrow 3x(2x - 3) \\ \text{Subtract} & + 4x - 6 & \\ & \underline{4x - 6} & \longleftarrow 2(2x - 3) \\ \text{Subtract} & 0 \text{ Remainder} & \end{array}$$

The quotient is $(3x + 2)$ If the solution is correct, the product of the quotient and divisor should be equal to the dividend

$$\begin{aligned} (3x + 2)(2x - 3) &= 3x(2x - 3) + 2(2x - 3) \\ &= 6x^2 - 9x + 4x - 6 = 6x^2 - 5x - 6 \end{aligned}$$

Whenever there is a break in the consecutive order of the exponents of a polynomial that appears in the dividend, the missing terms are included with zero coefficients Thus an expression of the form $5x^4 - 2x^2 - 10$ would be replaced with an equivalent polynomial of the form

$$5x^4 + 0x^3 - 2x^2 + 0x - 10$$

The next example may tend to clarify the need for this technique

EXAMPLE 7 Q

Divide $(6y^4 - 3y + 18)$ by $(-3y^2 + 6y - 9)$

Solution:

First, replace the original dividend with an equivalent polynomial containing the missing terms.

$$(6y^4 - 3y + 18) \text{ is equivalent to}$$
$$(6y^4 + 0y^3 + 0y^2 - 3y + 18)$$

Next, proceed with the division using the equivalent polynomial as the dividend.

$$\begin{array}{lrl}
 & -2y^2 - 4y - 2 & \\
-3y^2 + 6y - 9 & \overline{)\,6y^4 + 0y^3 + 0y^2 - 3y + 18} & \\
 & \underline{6y^4 - 12y^3 + 18y^2} & \longleftarrow (-2y^2)(-3y^2 + 6y - 9) \\
\text{Subtract} & +12y^3 - 18y^2 - 3y & \\
 & \underline{+12y^3 - 24y^2 + 36y} & \longleftarrow (-4y)(-3y^2 + 6y - 9) \\
\text{Subtract} & + 6y^2 - 39y + 18 & \\
 & \underline{+ 6y^2 - 12y + 18} & \longleftarrow (-2)(-3y^2 + 6y - 9) \\
\text{Subtract} & -27y & \text{Remainder}
\end{array}$$

The answer is written as Quotient + Remainder ÷ Divisor:

$$-2y^2 - 4y - 2 + \frac{-27y}{-3y^2 + 6y - 9}$$

Checking:

Divisor (Quotient + Remainder ÷ Divisor) = Dividend

$$(-3y^2 + 6y - 9)\left[(-2y^2 - 4y - 2) + \left(\frac{-27y}{-3y^2 + 6y - 9}\right)\right]$$

$$\begin{array}{r}
-3y^2 + 6y - 9 \\
\underline{-2y^2 - 4y - 2} \\
6y^2 - 12y + 18 \\
12y^3 - 24y^2 + 36y \\
\underline{6y^4 - 12y^3 + 18y^2} \\
(6y^4 + 0y^3 + 0y^2 + 24y + 18) + (-27y) = 6y^4 - 3y + 18
\end{array}$$

Although it is customary to introduce zero coefficients to complete a polynomial appearing as the dividend, the rule does not extend to the divisor. The *only restriction concerning the divisor is that the terms appear in the* same order as those in the dividend.

$$3x^3 - 2x + 3\,\overline{)\,3x^4 - 5x^3 + 0x^2 - 3x + 9}$$

rather than

$$3 - 2x + 3x^3\,\overline{)\,3x^4 - 5x^3 + 0x^2 - 3x + 9}$$

Perform the indicated operations

1. $(x^3)(x^7)$

2. $\dfrac{x^7}{x^3}$

3. $(2x^3)(6x^7)$

4. $\dfrac{6x^7}{2x^3}$

5. $(x^3)^7$

6. $(x^7)^3$

7. $(x^2y^3)^4$

8 $\left(\dfrac{x^7}{x^3}\right)^3$

9. $(3xy)(-2xy^2)$

10. $(6x^2y) - (-3xy)$

11. $(12ax)(3a - 5x)$

12 $(-7ay)(5ax)(-2xy)$

13. $(-2ax^2y)^3$

14 $(-2y)(x^2 - 5y^2)$

15 $\dfrac{18a^2b^3c}{-3ab^2c}$

16 $\dfrac{-36x^2yz^3}{-4x^2z^4}$

17. $(3x^2 - 6x) - (3x^2)$

18. $(21y^2 - 7) - (-7y)$

19. $(2x + 3)(x - 5)$

20. $(2x^2 - 7x - 15) - (x - 5)$

21. $(3x^2 - 2xy + 9y^2)(2xy)$

22. $(2a - 4b)(2a + 4b)$

23. $(7y^2 - 12xy + 3x^2)(3y - 2x)$

24. $(6x^2 - 5y^2)(3x + y)$

25. $(2x + 5y)^2$

26. $(2 - 3x)(3 + 6xy - 5y^2)$

27. $(5x + 2y)(x - 3y)(2x - 5y)$

28. $(x - y)(x^2 + xy + y^2)$

29. $(x^2 - 2xy + y^2)(x^2 + 2xy + y^2)$

30. $(2a^2 + 3ab + 5b^2)(a^2 - ab + b^2)$

31. $(x^2 - 2x + 1) - (x - 1)$

32. $(x^2 - 2xy + y^2) - (x - y)$

33. $(x^2 - y^2) - (x - y)$

34. $(10a^3 + 19a^2 - 9) - (2a + 3)$

35. $(9x^5 - 3x^4 + x^3 - 14x^2 - 17x - 4) - (3x + 1)$

36. $(x^3 - 3x^2y + 3xy^2 - y^3) - (x - y)$

37. $(4x^4 - 16) - (x^2 - 2)$

38. $(6a^5 - 4a^4 - 7a^3 + 21a^2 - 13a + 5) - (3a^2 - 2a + 1)$

39. $(y^4 - 81) - (y^2 - 9)$

40. $(4y^4 - 81) - (2y^2 + 9)$

41. $(8y^3 - 36y^2 + 54y - 27) - (2y - 3)$

42 $(4a^3 + 1 - 8a^2 + 3a^4) - (+2a - 1 + a^2)$

43. $(16a^8 - 81b^8) - (4a^4 + 9b^4)$

A cable used to support a suspended load, fixed at both ends, will form a curve that very closely approximates a parabola The length of this cable between supports can be determined for design purposes by the equation

$$L = l\left[1 + \frac{8}{3}\left(\frac{d}{l}\right)^2\right]$$

where

d is the sag of the cable in feet
l is the distance between supports in feet (span)
L is the length of the cable in feet (Fig. 7-3)

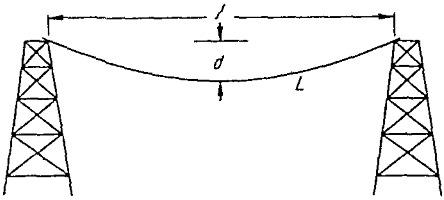


Figure 7-3

44. Find the length of a suspension cable in which the span of the bridge, l, $= 500.0$ ft and the sag in the cable, d, $= 30.0$ ft.

45. A power line sags 5.0 ft while being supported at points 150 ft apart. Find the developed length of the line.

46. Cables of a suspension bridge are 1500.0 ft apart. The maximum sag owing to loading is 75.0 ft. Find the length of the cables.

The equation $s = \frac{1}{2}g(2t - 1)$ is used to find the distance that a body falls (freely) during any given second.

s is the distance of fall in feet
g is the pull of gravity, taken as 32.2 ft/sec^2
t is the time in seconds

47. Find the distance a body will fall during the fifth second of travel ($t = 5$).

48. Find the distance a body will fall after 5 sec of flight.

7-9 SPECIAL PRODUCTS

Certain type products appear quite frequently in science and mathematics and are termed *special products*. The technician will use some of these consistently whereas others will appear sparingly. Each of the expressions can be multiplied out in the usual manner without ever referring to them as special products. Once the type product is recognized, however, expansion will be expedited and errors, hopefully, minimized. Furthermore, the materials studied in this section will serve as an introduction to other topics in algebra, soon to follow.

The general form of the special product will be listed first, followed by a brief description or definition concluding with an illustrative example.

1. $a(x + y) = ax + ay$ **Distributive Law**
$-3(2x - 6y) = -6x + 18y$

2. $(x + y)(x - y) = x^2 - y^2$ **Difference of two squares**
$(2a + 5b)(2a - 5b) = (2a)^2 - (5b)^2 = 4a^2 - 25b^2$

3. $(x + y)^2 = x^2 + 2xy + y^2$ **Perfect square trinomial**
$(3x + 7y)^2 = (3x)^2 + 2(3x)(7y) + (7y)^2 = 9x^2 + 42xy + 49y^2$

4. $(x - y)^2 = x^2 - 2xy + y^2$ **Perfect square trinomial**
$(3x - 7y)^2 = (3x)^2 - 2(3x)(7y) + (7y)^2 = 9x^2 - 42xy + 49y^2$

5. $(x + a)(x + b) = x^2 + (a + b)x + ab$ **Trinomial**
$(y - 5)(y + 3) = (y)^2 + (-5 + 3)y + (-5)(+3) = y^2 - 2y - 15$

6. $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$ **General trinomial**
$$(3x + 2)(2x - 3) = (3)(2)x^2 + [(3)(-3) + (2)(2)]x + (2)(-3)$$
$$= 6x^2 + [-9 + 4]x - 6 = 6x^2 - 5x - 6$$

7. $(x + y)(x^2 - xy + y^2) = x^3 + y^3$ **Sum of two cubes**
$(2x + 3y)(4x^2 - 6xy + 9y^2) = 8x^3 + 27y^3$

8. $(x - y)(x^2 + xy + y^2) = x^3 - y^3$ **Difference of two cubes**
$(2x - 3y)(4x^2 + 6xy + 9y^2) = 8x^3 - 27y^3$

9. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ **Square of a trinomial**
$$(3x - 2y + 4z)^2 = (3x)^2 + (-2y)^2 + (4z)^2 + 2(3x)(-2y)$$
$$+ 2(3x)(4z) + 2(-2y)(4z)$$
$$= 9x^2 + 4y^2 + 16z^2 - 12xy + 24xz - 16yz$$

10. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ **Cube of a binomial**
$$(4x + 3y)^3 = (4x)^3 + 3(4x)^2(3y) + 3(4x)(3y)^2 + (3y)^3$$
$$= 64x^3 + 144x^2y + 108xy^2 + 27y^3$$

11. $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$ **Cube of a binomial**
$(4x - 3y)^3 = 64x^3 - 144x^2y + 108xy^2 - 27y^3$

EXERCISES 7-6

Each of the products in the following exercises can be associated with a special product listed previously The object is to complete these products by inspection through association with the general forms Some expressions may not appear to be linked with a special product, in this event, the problem should be completed by other procedures established for multiplying polynomials

1. $2(3 - x)$
2. $2(3 - x)x$
3. $-5x(2 + 3x)$
4. $-3(a - 3y)$
5. $(x - 1)(x + 1)$
6. $(2 - y)(2 + y)$
7. $(3x - 2y)(3x + 2y)$
8. $(3x - 2y)^2$
9. $(3x + 2y)^2$
10. $(3x - 2y)(3x - 2y)$
11. $(3 + 5x)(9 - 15x + 25x^2)$
12. $(3 - 5x)(9 + 15x + 25x^2)$
13. $5\left(\frac{1}{2}x - \frac{3}{2}y\right)^2$
14. $(7a - b)(a + 7b)$

15. $(-10x + 3y)(-2x - 3y)$

16. $(x - 2y^2)(2x + y^2)$

17. $3(2 - 3y)(3 + 2y)$

18. $(x - 4)(x^2 + 4x + 16)$

19. $(x + 4y)(x^2 - 4xy + 16y^2)$

20. $(2x + 1)^2(2x - 1)^2$

21. $(x - 3y + 2z)^2$

22. $(x - 3)^3$

23. $(2x + 3y)^3$

24. $[(2x - 3) + 2y][(2x - 3) - 2y]$

25. $[(2x - 3) - 2y]^2$

26. $(-3z + 4x - 5y)^2$

27. $(3x - \frac{7}{3}y)(6x + \frac{5}{3}y)$

28. $(3x + 4y)^3$

29. $(3x - y - 2z)(3x - y + 2z)$

30. $(3x - y - 2z)(3x + y + 2z)$

7-10 FACTORING

Factoring may be considered as a process that is the inverse of finding special products or products in general. The principle of factoring leads to a procedure for solving certain mathematical equations as well as engineering formulas. Factoring is also a useful technique associated with various other mathematical operations.

When two or more expressions are multiplied together, they form a product. Each of the **terms** involved in the multiplication is called a **factor** of the **product.**

(2) (3) (5) = 30; here, 2,3, and 5 are considered factors of 30. Furthermore, 2 and 5, 5 and 6, and 3 and 10 can also be referred to as factors of 30. They are not, however, prime factors and the technician's concern will usually be with obtaining prime factors.

In the expression $3x(2 - 3y) = 6x - 9xy$, the quantities $3x$ and $(2 - 3y)$ are considered factors of the binomial $6x - 9xy$. Furthermore, $3x$ has the factors 3 and x. Thus, the prime factors of $(6x - 9xy)$ are 3, x, and $(2 - 3y)$. Normally, monomial factors are not expressed as prime factors. Hence, $3x$ would not likely be written as $(3)(x)$. Fundamentally, factoring is the process of finding several expressions whose product is equal to a given polynomial. An expression is **completely factored** when it is **expressed** in terms of the products of **prime factors.**

7-11 FACTORING BY INSPECTION

There are several procedures, structured as well as intuitive, that are used to find the factors of a given algebraic expression. One of these is called **factoring by inspection.**

This procedure implies that the given polynomial contains a common monomial factor that is easily recognized, or that the given expression is associated with a known special product. Once a factor has been identified, a second appears as the quotient of the given polynomial and the first factor.

$$\text{(quotient)} \times \text{(divisor)} = \text{dividend}$$

In terms of the present discussion this can be viewed as

$$(\text{second factor}) \times (\text{first factor}) = \text{polynomial}$$

or

$$\frac{(\text{polynomial})}{(\text{first factor})} = (\text{second factor})$$

The procedure of dividing the first factor into the given polynomial to find another factor is referred to as *factoring out an expression common to all of the terms of the original expression*

Several illustrations may help to develop the concept of factoring

EXAMPLE 7 R

Factor $3ax^2 - 6ay + 18az$

Solution

After brief study, it becomes apparent that $3a$ is the highest common multiple of every term in the given polynomial

$$(3a)(x^2) - 2(3a)y + 6(3a)z$$

or

$3a(x^2 - 2y + 6z)$ which is the given expression factored completely

$$3ax^2 - 6ay + 18\,az = 3a(x \;\; - 2y + 6z)$$

Factoring out $3a$ is identical to dividing $3ax^2 - 6ay + 18az$ by $3a$ whereby the quotient becomes the second factor

$$\begin{array}{r@{}l}
 & \;\; x^2 - 2y \;\; + 6z \\ \cline{2-2}
3a\,) & \; 3ax^2 - 6ay + 18az \\
 & \; \underline{3ax^2} \\
 & \;\;\; 0 \;\; - \underline{6ay} \\
 & \qquad\quad\;\; 0 + \underline{18az} \\
 & \qquad\qquad\qquad 0
\end{array}$$

EXAMPLE 7 S

Find the factors of $4x^2 - 81y^2$

Solution

The binomial has all the appearances of the special product termed the **difference of two squares**

$$\text{In general form, } x^2 - y^2 = (x - y)(x + y)$$

A number n is considered to be a perfect square if there exists another number m, such that

$$\boldsymbol{n = m^2}, \textbf{ where } \boldsymbol{m} \textbf{ is a rational number.}$$

Notice that the first term of the binomial is the square of the first term in each factor, and the second term of the binomial is the square of the second term in each factor.

In the problem under discussion,

$4x^2 = (2x)(2x) = (2x)^2$, where $4x^2$ is called the square of $2x$, and
$81y^2 = (9y)(9y) = (9y)^2$, where $81y^2$ is called the square of $9y$.
Hence, $4x^2 - 81y^2 = (2x - 9y)(2x + 9y)$

EXAMPLE 7-T:

Factor the trinomial $x^2 - 6x + 9$

Solution:

While studying a trinomial for possible factors, it is recommended that initial consideration be directed to the first and third terms. If **both** of these **terms** turn out to be **perfect squares**, it's conceivable that the expression could be associated with these special products:

$$(x - y)^2 = x^2 - 2xy + y^2$$

or

$$(x + y)^2 = x^2 + 2xy + y^2,$$

both of which are referred to as **perfect square trinomials.**

Returning to the example, $x^2 - 6x + 9$, it should be noted that

$x^2 = (x)(x)$ and $9 = (3)(3)$. Thus, x^2 and 9 are perfect squares.

Along with these conditions, a second criterion involving the middle term of the trinomial must be satisfied. If the **middle term** is equal to **twice** the **cross-product** of the terms of the binomial factors, the trinomial represents a perfect square trinomial. Using the general form;

$$(x - y)(x - y) = x^2 - 2xy + y^2$$

$$-xy$$

$$\begin{array}{r} -xy \\ \underline{-xy} \\ -2xy \end{array}$$

the indicated multiplications are referred to as cross-products

It appears that $x^2 - 6x + 9$ meets the criterion defining a perfect square trinomial; therefore,

$$x^2 - 6x + 9 = (x - 3)(x - 3) = (x - 3)^2$$

$$-3x$$

$$\begin{array}{r} -3x \\ \underline{-3x} \\ -6x \end{array}$$

An error that seems to present itself occasionally is failure to recognize the distinction between the difference of two squares and the perfect square trinomial

Difference of two squares $x^2 - 9 = (x - 3)(x + 3)$
Perfect trinomial square $(x - 3)^2 = (x - 3)(x - 3)$
Thus, $x^2 - 9 \neq (x - 3)^2$

Factoring of the general trinomial $ax^2 + bx + c$ will be studied later, presently the discussion will continue with other special products

EXAMPLE 7 U

Find the factors of $8a^3 + 125b^3$

Solution

This should be recognized as the sum of two cubes where

$$8a^3 = (2a)(2a)(2a) \quad \text{and} \quad 125b^3 = (5b)(5b)(5b)$$

In general form (sum of two cubes),

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

Thus,

$$\begin{aligned} 8a^3 + 125b^3 &= (2a + 5b)[(2a)^2 - (2a)(5b) + (5b)^2] \\ &= (2a + 5b)(4a^2 - 10ab + 25b^2) \end{aligned}$$

EXAMPLE 7-V

Factor $4x^2y^4 - 64x^2z^4$

Solution

This example is designed to emphasize two conditions involving factoring Initially, the original expression should be studied for possible common multiples, such as $4x^2$ in the binomial $4x^2y^4 - 64x^2z^4$

Whenever this situation is evident, the common monomial should be factored out in the first step Hence,

$$4x^2y^4 - 64x^2z^4 = 4x^2(y^4 - 16z^4)$$

and the original expression is simplified somewhat which was the designed intent The factor $(y^4 - 16z^4)$ is recognized as the difference of two squares and resolved accordingly Thus,

$$4x^2y^4 - 64x^2z^4 = 4x^2(y^4 - 16z^4) = 4x^2(y^2 - 4z^2)(y^2 + 4z^2)$$

This leads to the second condition which is to caution against the tendency to fall short of a completely factored expression A further study of

the factor $(y^2 - 4z^2)$ suggests that this binomial also represents the difference of two squares.

Therefore,

$$y^2 - 4z^2 = (y - 2z)(y + 2z)$$

and

$$4x^2y^4 - 64x^2z^4 = 4x^2(y - 2z)(y + 2z)(y^2 + 4z^2),$$

wherein the last expression represents all the prime factors of $4x^2y^4 - 64x^2z^4$.

A binomial of the form $y^2 + 4z^2$, except for monomial factors, cannot be factored further.

The technique of factoring, presently, depends on the ability to recognize special products. Both of these topics complement each other and the understanding of one strengthens mastery of the other.

EXERCISES 7-7

Factor completely (Refer to the list of special products, Sec. 7-9).

1. $7ax - 21bx$
2. $3ax + 24bx - 15cx$
3. $5abc - 20bc - ac$
4. $4x^2y^2 - 81xy - 4y$
5. $(5 - 2b)x - (5 - 2b)y$
6. $15a(x - 3y) + 20b(x - 3y) - 45(x - 3y)$
7. $x^2 - 25y^2$
8. $y^2 - 25x^2$
9. $16x^2 - 9y^2$
10. $(x + 2)^2 - 81y^2$
11. $(2x - 1)^2 - (3y + 1)^2$
12. $(x + y)^2 - (x - y)^2$
13. $49a^2 - 144b^4$
14. $169c^2 - 1$
15. $x^2 - 2x + 1$
16. $x^2 + 2x + 1$
17. $1 - 2y + y^2$
18. $4 + 4y + y^2$
19. $x^2 - 24x + 144$
20. $25 + 10a + a^2$
21. $4x^2 - 20xy + 25y^2$
22. $9x^2 + 42xy + 49y^2$
23. $x^3 - 512$
24. $x^3 + 512$
25. $125 - 27y^3$
26. $216a^6 - 27b^3$
27. $216a^3 - 27a^3$
28. $3x^3 + 24y^3$
29. $(2x - 5y)^3 - (2x - 5y)$
30. $a^3 - 3a^2b + 3ab^2 - b^3$
31. $8x^3 + 36x^2y + 54xy^2 + 81y^3$
32. $125x^3 - 300x^2y + 240xy^2 - 64y^3$
33. $4x^2 + 9y^2 + z^2 - 12xy + 4xz - 6yz$
34. $16x^2y^2 - 25x^2z^2$
35. $a^{2x} - 2a^x + 1$
36. $a^6 - 1$
37. $x^4 - y^4$

An expression of the form $ax^2 + bx + c$ is referred to as the *general trinomial* or trinomial in general form wherein a, b, and c are rational numbers.

When the coefficient of the second-degree term is equal to 1, the trinomial can usually be factored by inspection, if at all factorable. For the condition wherein $a = 1$, the procedure basically centers around finding two quantities whose product is equal to c and whose sum is equal to b. This principle will be applied to the following trinomial:

$$x^2 + 5x + 6, \text{ where } a = 1,\ b = 5, \text{ and } c = 6$$

It is evident that

$$(2)(3) = 6 \quad (c)$$

and

$$2 + 3 = 5 \quad (b)$$

Thus,

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

A technique for multiplying two binomials by inspection is given below:

$$\begin{array}{l} \quad x^2 \qquad 6 \\ (x + 2)(x + 3) = x^2 + 5x + 6, \text{ which is similar to} \\ \qquad +2x \\ \\ \qquad \underline{+3x} \\ \qquad +5x \end{array}$$

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd \qquad \text{(6—Sec. 7-9)}$$

Notice in both cases that the product containing the first term of each binomial defines the first term of the expansion. Furthermore, the product of the second terms leads to the last term of the expansion. Finally, the middle term of the trinomial is equal to the sum of the cross-products.

In general, the factors of $x^2 + bx + c$ are $(x + p)(x + q)$, where $(p)(q) = c$ and $(p + q) = b$.

The signs of the terms in the trinomial $x^2 + 5x + 6$ will be varied to illustrate the effect on the factors.

$$\begin{aligned} x^2 + 5x + 6 &= (x + 2)(x + 3) \\ x^2 - 5x + 6 &= (x - 2)(x - 3) \\ x^2 - 5x - 6 &= (x + 1)(x - 6) \\ -x^2 + 5x + 6 &= -(x^2 - 5x - 6) = -(x + 1)(x - 6) \end{aligned}$$

A suggested procedure for factoring the general trinomial $ax^2 + bx + c$ is listed below in the form of guidelines.

1. Factor out common multiples;
2. Find all possible factors of ax^2 (or a);
3. Find all possible factors of c; and,
4. Select that combination of factors of a and c such that the sum of their cross-products will be equal to b.

Summarizing,

$$ax^2 + bx + c = (rx + p)(sx + q)$$

where

$$(r)(s) = a$$
$$(p)(q) = c$$

and

$$(p)(s) + (r)(q) = b$$

EXAMPLE 7-W:

Factor $6x^2 - 23x + 20$.

Solution:

Several patterns have been developed whereby the various trial factors are put in combination and tested. Two such arrangements will be demonstrated. Regardless of the technique, the problem resolves itself to finding multiples of a and c whose sum is equal to the middle term b.

$a = 6$	$c = 20$
Factors are (3) (2) and (6) (1)	Factors are (20) (1), (10) (2), (5) (4), (−20) (−1), (−10) (−2), and (−5) (−4)

A pair of factors of 6 must be combined with a pair of factors of 20 such that the sum of the cross-products is equal to −23.

First trial:	$(3x - 20)(2x - 1) = 6x^2 - 43x + 20$
Second trial:	$(3x - 1)(2x - 20) = 6x^2 - 62x + 20$
Third trial:	$(3x - 10)(2x - 2) = 6x^2 - 26x + 20$
Fourth trial:	$(3x - 2)(2x - 10) = 6x^2 - 34x + 20$
Fifth trial:	$(6x - 10)(x - 2) = 6x^2 - 22x + 20$
Sixth trial:	$(3x - 5)(2x - 4) = 6x^2 - 22x + 20$
Finally:	$\mathbf{(3x - 4)(2x - 5) = 6x^2 - 23x + 20}$

Obviously some combinations eliminate themselves by inspection, however, all were carried out as a matter of illustration

The same problem will be solved using a different method of grouping the various combinations Only the numerical quantities are used Once the proper set has been firmed up, however, the literal portion will be incorporated and the solution completed

$$6x^2 - 23x + 20$$

$a = 6$	$c = 20$	$b = -23$
6	-10	
		Sum of cross-products $(-10) + (-12) = -22$
1	-2	
3	-5	
		Sum of cross-products $(-12) + (-10) = -22$
2	-4	
3	-4	
		Sum of cross products $(-8) + (-15) = -23$
2	-5	

Thus, a workable combination has been realized that leads to the final step of arranging the terms in the respective factors The diagram suggests this alignment

$$6x^2 - 23x + 20 = (3x - 4)(2x - 5)$$

EXAMPLE 7-X

Find the factors of $40x^2 - 3x - 28$

Solution

In some convenient manner, list the multiples of 40 and -28 (integers only)

$a = 40$	$b = -3$	$c = -28$
(40) (1)		$(-28)(1)$, $(-1)(28)$
(20) (2)		$(-7)(4)$, $(-4)(7)$
(10) (4)		$(-14)(2)$, $(-2)(14)$
(8) (5)		

Some of these combinations can be eliminated from serious considerat on by inspection For example, it hardly seems likely that (40) (1) could very well combine with any set on the right to come up with a cross product sum of -3 The same holds true for $(-28)(1)$ or $(-1)(28)$ A few trials and further study seem to indicate that the solution rests with this combination

$$40x^2 \qquad -28$$

$$(5x + 4)(8x - 7) = 40x^2 - 3x - 28$$

$$+32x$$

$$\begin{array}{r} -35x \\ \hline -3x \end{array}$$

The process of factoring becomes more orderly as the technician, through experience, develops the sense of juggling the various combinations, by inspection, until the proper arrangement suggests itself.

7-13 FACTORING BY GROUPING

Some polynomials of the form $ax + ay + bx + by$ can be factored by proper grouping of terms and removal of a common monomial multiple. The procedure after this initial step is identical to those previously established.

EXAMPLE 7-Y:

Factor $3xy + 2x - 27y - 18$.

Solution:

A study of the expression indicates that the first two terms have a common factor and the last two terms also have a common factor, x and -9, respectively. Thus,

$$3xy + 2x - 27y - 18 = x(3y + 2) - 9(3y + 2)$$

Now it becomes evident that there exists a common binomial factor; namely $(3y + 2)$, which can be treated accordingly.

$$x(3y + 2) - 9(3y + 2) = (3y + 2)(x - 9)$$

Thus,

$$3xy + 2x - 27y - 18 = (3y + 2)(x - 9)$$

EXAMPLE 7-Z:

Factor the expression $x^2y - 2 - y + 2x^2$.

Solution:

In the present form, no two consecutive terms suggest a possible monomial factor. The situation may change if the terms are rearranged, however. Hence,

$$x^2y - 2 - y + 2x^2 = x^2y - y + 2x^2 - 2 = x^2y + 2x^2 - y - 2$$

Both of the equivalent polynomials suggest that monomial factors are

present *The re-arrangement of terms, leading to monomial factors, is referred to as proper grouping*

Both the equivalent polynomials will now be factored

$$x^2y - y + 2x^2 - 2 = y(x^2 - 1) + 2(x^2 - 1) = (x^2 - 1)(y + 2)$$
$$= (x - 1)(x + 1)(y + 2)$$
$$x^2y + 2x^2 - y - 2 = x^2(y + 2) - 1(y + 2) = (x^2 - 1)(y + 2)$$
$$= (x - 1)(x + 1)(y + 2)$$

Notice that the results are identical, which indicates that *proper grouping* is not limited to one form

EXERCISES 7-8

Factor the following polynomials

1. $x^2 - 5x + 6$

2. $x^2 - 5x - 6$

3. $x^2 - 3x + 2$

4. $y^2 + 7y + 10$

5. $y^2 + y - 30$

6. $x^2 + x - 56$

7. $12a^2 + a - 6$

8. $8x^2 + 6xy - 9y^2$

9 $15x^2 + 16x - 15$

10. $28y^2 - 65y + 28$

11. $30x^2 + 32x - 30$

12. $28y^2 - 33y - 28$

13. $36y^2 + 25y - 25$

14. $9x^2y^2 - 6xy + 1$

15. $30x^2 + 11xy - 30y^2$

16. $6x^2 - 21x + 18$

17. $16x^2 - 18xy - 9y^2$

18. $3 - 18xy + 27x^2y^2$

19. $25x^2 - 20xy + 4y^2$

20. $17x^2 + 68x + 51$

21. $4x^2 + 2x + \frac{1}{4}$

22. $9x^2 - 2x + \frac{1}{9}$

23. $132y^2 + y - 156$

24. $8a^2 - 2a - 105$

25. $3xy - 4 + 2x - 6y$

26. $x^2z + xz^2 - z - x$

27. $1 + xy + y + x$

28. $2x^2y - 4xy + 8x - 16$

29. $2ab + 6bc - abx - 3bcx$

30. $2x^2y - 9ay - 6xy^2 + 3ax$

31. $-16ay - 12ax + 3ab^2x + 4ab^2y$

32. $15ax^2 - 28ab - 20ax + 21abx$

33. $y^3 + 8y^2 - 32y - 64$

34. $x^2 + y^2 - x + y - 2xy$

35. $4x^2 - 3 - 2x - 9$

36. $3mn + 18am + 2n^2 + 12an$

REVIEW EXERCISES 7-9

Simplify the following expressions (Ex 1-15)

1. $-2x[3 - 2(x + 2)] + 2x[2 - (3x - 2)]$

2. $-3\{4 - 2[3 - (5 + 2)] - 7(5 - 2)\} + 10$

3. $(2ax - 3by) + (4ax + 3by)$

4. $(2bx - 3ay) - (3bx + 2ay)$

5. $(3x^2 + 5x - 6) + (-2x^2 + 3x + 6)$

6. $(15ab - 7bc + b) - (13ab - 5bc - 2b)$

7. $-(3 - 5x + 3y) - (4y + 7x)$

8. $(3x + y) - [2 + 3(x - 2y) + 6x]$

9. $(5x - 2y) - 2x(-4) + 3(x - 5y) - 3(x - 5y)$

10. $(2a^2)(3a^3)$

11. $(2a^2)^3$

12. $\dfrac{16a^4}{8a^5}$

13. $(3a^2)^2\left(\dfrac{4a}{3a}\right)^2$

14. $4(a^5)^2 + 3(a^2)^5 - [6(a^4)(a^6)]$

15. $2(x^2)^3 + (3x^3)(2x^3) - 5\left(\dfrac{2x^5}{x^2}\right)^2 + 2x^6$

Perform the indicated operations (Ex. 16-26).

16. $(2x - 7y)(3x + 4y)$

17. $(3a - 5b)(3a + 5b)$

18. $(3x + 2y - 3)(3x + 2y - 3)$

19. $(2x - 3y)(4x^2 + 6xy + 9y^2)$

20. $(5x + 2y)(25x^2 - 10xy + 4y^2)$

21. $(4x^2 - 12xy + 9y^2) \div (-3y + 2x)$

22. $(9x^2y^2 - 100) \div (3xy - 10)$

23. $(27x^3 - 135x^2y + 225xy^2 - 125y^3) \div (3x - 5y)$

24. $(3x^2 - 7x + 15y - 5xy - 6) \div (x - 3)$

25. $(7x + 4y - 6z)(7x + 4y + 6z) \div (7x - 6z + 4y)$

26. $(4x - 5y)^3$ (expand)

Factor completely (Ex. 27-36).

27. $7x^2 - 51x + 14$

28. $25x^2 - 130xy + 169y^2$

29. $16x^2 - 49y^2$

30. $15x^4 + 4x^2y - 4y^2$

31. $60y^2 - 135x^2$

32. $27x^3 - 81x^2y + 81xy^2 - 27y^3$

33. $3ax - 10y + 6x - 5ay$

34. $12ax^2y + 18by - 27ay - 8bx^2y$

35. $27x^3 - 343y^6$

36. $25x^2 + 9y^2 + 4z^2 - 30xy - 20xz + 12yz$

37. The correction, C_T, in length owing to temperature change of a steel surveyor's tape can be determined by the formula

$C_T = 6.5 \times 10^{-6}(T_2 - T_1)L$, where
T_2 is the temperature (°F) when the length, L (ft), was measured.
T_1 is the temperature at which the tape was standardized, taken as 68°F.

Find the change in length of tape when a line 635 20 ft was measured when the temperature was 85°F

38. The volume, V, of excavation can be approximated by the formula

$$V = \frac{L}{6}(A_1 + A_2 + 4A_m)$$

where

L is the horizontal distance (feet) between A_1 and A_2
A_1 is the cross-section area of one end (square feet)
A_2 is the cross-section of the other end, and
A_m is the area mid-way between A_1 and A_2

Find the volume of dirt taken out of an excavation if

$L = 250$ ft, $A_1 = 1{,}275$ ft², $A_2 = 1{,}325$ ft², and
$A_m = 1{,}300$ ft²

39. The maximum deflection, d, of a beam supported at both ends with a concentrated load, P, at any point along the beam is given by the formula

$$d = \frac{1}{16}\frac{Pb}{EI}\left(L^2 - \frac{4}{3}b^2\right)$$

where

P is the load (pounds) at a distance b (inches) from the nearest support
L is the length of the beam (inches)
E is the modulus of elasticity, and
I is the moment of inertia

Find the maximum deflection of a simple beam with fixed ends if

$P = 6{,}000$ lb, $L = 16\ 0$ ft, $E = 30 \times 10^6$ lb/in²
$I = 76\ 0$ in⁴, and $b = 60\ 0$ in

40. In a certain series-parallel network, the resistance between two terminals, x and y, is given by the formula

$$R_{xy} = R_x + \frac{(R_y + R_a)(R_b + R_c)}{R_y + R_a + R_b + R_c}$$

Find R_{xy} if $R_x = 2\ 5$ ohms, $R_y = 3\ 2$ ohms, $R_a = 1\ 1$ ohms,
$R_b = 0\ 8$ ohm and $R_c = 1\ 5$ ohms

CHAPTER 8

Algebraic Fractions

The quotient of two algebraic expressions is referred to as an *algebraic fraction*.

$$\frac{a}{b}, \frac{1}{a+3b}, \frac{a+3b}{4c}, \frac{3x^2y(7x-20y)}{8xy^2(7x-20y)^2}, \ldots .$$

In the fraction $(a+3b)/4c$, the term $(a+3b)$ is called the *numerator* or *dividend* and $(4c)$, is called the *denominator* or *divisor.*

Algebraic fractions can be combined in the various ways described by the fundamental laws of arithmetic. An algebraic fraction is *undefined* or *indeterminate* if the *denominator is equal to zero.* If the *numerator is equal to zero*, the *value of the fraction is equal to zero.* In the fraction $(a+3b)/4c$, let $a=-3$, $b=1$, and $c=2$. It follows that $(-3+3)/8=0/8=0$. Furthermore, if $a=1$, $b=2$, and $c=0$, then the value of the fraction becomes $(1+6)/0=7/0$, which is *undefined* or *indeterminate.* What this really means is that the fraction cannot be assigned a numerical value without other information.

8-1 PROPERTIES OF FRACTIONS

The fundamental properties of fractions (Sec. 2-2) also apply to algebraic fractions. Thus, the numerator and denominator of an algebraic fraction can be multiplied or divided by the same algebraic expression without changing the value of the fraction. The algebraic expression, however, must be other than zero.

$$\frac{a}{b}=\frac{2(a)}{2(b)}; \frac{a+b}{c-d}=\frac{(6x)(a+b)}{(6x)(c-d)}, \ldots .$$

It is the application of the fundamental principle or properties of fractions that leads to an equivalent fraction.

Equivalent fractions are fractions that are identical, that is, they have the same value

$$\frac{3a}{4a} = \frac{15a}{20a},\ \frac{6a - 6b}{a^2 - b^2} = \frac{6(a - b)}{(a - b)(a + b)},$$
$$\frac{(2x - 1)}{(3x + 4y)} = \frac{(2x - 1)(2x + 1)}{(3x + 4y)(2x + 1)}$$

where, of course, no value assigned to the literal number produces a denominator that is equal to zero

The principle reasons for becoming involved with equivalent fractions is to

1 find a common denominator so that the fractions can be added algebraically

$$\frac{2a}{5b} + \frac{a}{10b} = \frac{4a}{10b} + \frac{a}{10b} = \frac{5a}{10b} = \frac{a}{2b}$$

2 reduce fractions to lowest terms

$$\frac{6a - 6b}{a^2 - b^2} = \frac{6\cancel{(a - b)}}{\cancel{(a - b)}(a + b)} = \frac{6}{a + b}$$

where,

$$\frac{(a - b)}{(a - b)} = 1$$

One important principle that is associated with equivalent fractions and fractions in general is the *Law of Signs*

Fundamentally,

$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}$$

which states that the three fractions are equivalent

Furthermore,

$$\frac{a}{b} = \frac{(-1)(a)}{(-1)(b)} = (-1)\frac{(-1)(a)}{b} = (-1)\frac{a}{(-1)(b)} = \frac{a}{b}$$

Also,

$$\frac{(a - b)}{(b - c)} = \frac{(-1)(a - b)}{(-1)(b - c)} = \frac{-a + b}{-b + c} = \frac{b - a}{c - b}$$

Likewise,

$$\frac{(a - b)}{(b - c)} = (-1)\frac{(-1)(a - b)}{(b - c)} = (-1)\frac{(a - b)}{(-1)(b - c)}$$

As a stated rule, the concept just demonstrated can be written as follows:

The signs of the factors of a fraction can be changed an even number of times without changing the value of the fraction. Changing signs of the factors an odd number of times will change the value of the fraction.

The key word here is *factors*. In the expression $3a(2x - 3y)/5b(x + 4y)$, $3a$, $(2x - 3y)$, $5b$, and $(x + 4y)$ are *factors* whereas, $2x$, $-3y$, x, and $4y$ are *terms*.

The rule just mentioned refers to changes in sign, which means that the factors can be multiplied by (-1) or (-1) can be removed (factored out) from the various fractions.

Multiplying by (-1):

$$\frac{(2a - b)}{(3b - c)} = \frac{(-1)(2a - b)}{(-1)(3b - c)} = \frac{-2a + b}{-3b + c} = \frac{b - 2a}{c - 3b}$$

Factoring out (-1):

$$\frac{(2a - b)}{(3b - c)} = \frac{(-1)(-2a + b)}{(-1)(-3b + c)} = \frac{-2a + b}{-3b + c} = \frac{b - 2a}{c - 3b}$$

Using the last illustration, it can be demonstrated that $(2a - b)/(3b - c)$ and $(b - 2a)/(c - 3b)$ are equivalent fractions.

Let $a = 2$, $b = 5$, and $c = (-3)$. Then:

$$\frac{2a - b}{3b - c} = \frac{2(2) - (5)}{3(5) - (-3)} = \frac{4 - 5}{15 + 3} = \frac{-1}{18} = -\frac{1}{18}$$

and

$$\frac{b - 2a}{c - 3b} = \frac{5 - 2(2)}{(-3) - 3(5)} = \frac{5 - 4}{-3 - 15} = \frac{1}{-18} = -\frac{1}{18}$$

or

$$\frac{2a - b}{3b - c} = \frac{b - 2a}{c - 3b}$$

(even number of changes in sign, two)

EXAMPLE 8-A:

Given the fraction $-a/(b - c)$, write an equivalent fraction that does not contain a minus sign in the numerator.

Solution:

Multiply both numerator and denominator of the fraction by (-1).

$$\frac{-a}{b - c} = \frac{(-1)(-a)}{(-1)(b - c)} = \frac{a}{-b + c} = \frac{a}{c - b}$$

Alternate Solution

Multiply the fraction and the numerator by (-1)

$$\frac{-a}{b-c}=(-1)\frac{(-1)(-a)}{(b-c)}=(-1)\frac{(a)}{(b-c)}$$

which is equivalent to the fraction $a/(c-b)$

EXAMPLE 8 B

Simplify or reduce the given fraction to its lowest terms

$$\frac{4x^3-2x^2y-2xy^2}{x^2-y^2}$$

Solution

Start by factoring both numerator and denominator

$$\frac{4x^3-2x^2y-2xy^2}{x^2-y^2}=\frac{2x(2x^2-xy-y^2)}{(x-y)(x+y)}=\frac{2x(2x+y)\cancel{(x-y)}}{\cancel{(x-y)}(x+y)}$$

After canceling like factors the expression appears in simplified form as $2x(2x+y)/(x+y)$ (recall that $(x-y)/(x-y)=1$)

Several common errors involving the operation of canceling algebraic expressions keep persisting This happens most often because of the tendency to treat algebraic terms as though they were factors A few examples may help to clarify this important distinction

$$\frac{a}{a+b}\neq\frac{\cancel{a}}{\cancel{a}+b}\neq\frac{1}{b}$$

since a and b are terms of the fraction, not factors On the other hand,

$$\frac{2a}{2a^2+2ab}=\frac{\cancel{2a}}{\cancel{2a}(a+b)}=\frac{1}{a+b}$$

where it should be apparent that $2a$ and $(a+b)$ are factors **Identical factors appearing in the numerator and denominator of a fraction can be divided or canceled out** Actually, under these conditions the quotient of the factors is equal to 1 **Terms may not be canceled out unless they are contained in each element of the fraction**

This concept may be fortified by using an arithmetic example

$$\frac{2}{2+3}=\frac{2}{5},\ \text{rather than}\ \frac{2}{2+3}\neq\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{2}}+3}\neq\frac{1}{4}$$

In summary, referring to the fraction $2a/(2a+3ab)$, the literal number

a is contained in each element of the fraction. Therefore, it can be canceled out as indicated.

$$\frac{2a}{2a + 3ab} = \frac{2\cancel{a}}{2\cancel{a} + 3\cancel{a}b} = \frac{2}{2 + 3b},$$

Notice again that the quantity a is **common to all terms** of the fraction. Thus, it is a **common multiple** or **common factor** of both numerator and denominator.

Usually the above operation is carried out as

$$\frac{2a}{2a + 3ab} = \frac{2\cancel{(a)}}{\cancel{a}(2 + 3b)} = \frac{2}{2 + 3b}$$

where a is a common factor.

EXAMPLE 8-C:

Reduce

$$\frac{5ax + 15a}{3ax^2 - 27a}$$

to lowest terms.

Solution:

Factor numerator and denominator and cancel accordingly.

$$\frac{5ax + 15a}{3ax^2 - 27a} = \frac{5a(x + 3)}{3a(x^2 - 9)} = \frac{5\cancel{(x + 3)}}{3(x - 3)\cancel{(x + 3)}} = \frac{5}{3(x - 3)}$$

EXAMPLE 8-D:

Reduce to lowest terms.

$$\frac{24axz - 36axy - 40az + 60ay}{12bxz - 20bz + 30by - 18bxy}$$

To reduce a fraction to its lowest terms, factor both numerator and denominator and divide out or cancel like factors, where applicable.

Solution:

Factoring:

$$\frac{4a(6xz - 9xy - 10z + 15y)}{2b(6xz - 10z + 15y - 9xy)} = \frac{2a[3x(2z - 3y) - 5(2z - 3y)]}{b[2z(3x - 5) + 3y(5 - 3x)]}$$

The denominator could be factored further if the signs of the terms in the factor $(5 - 3x)$ could be changed to $(3x - 5)$. Recall that if the sign of a factor is changed an odd number of times, the sign of the factor (fraction) changes. Applying this principle, presently, leads to: $+3y(5 - 3x) = -3y(3x - 5)$. Substituting the equivalent factor in the preceding step leads to this expression:

$$\frac{2a[3x(2z-3y)-5(2z-3y)]}{b[2z(3x-5)-3y(3x-5)]}$$

Factoring further and canceling, leads to completion

$$\frac{2a[\cancel{(2z-3y)}\cancel{(3x-5)}]}{b[\cancel{(2z-3y)}\cancel{(3x-5)}]}=\frac{2a}{b}$$

Hence,

$$\frac{24axz-36axy-40az+60ay}{12bxz-20bz+30by-18bxy} \text{ reduces down to } \frac{2a}{b}$$

The various steps in the example just completed were carried out to demonstrate several techniques associated with factoring A point more important than the concepts of algebra will now be stressed Before attempting to solve any problem, study all conditions very carefully before plunging into a fixed mechanical routine Notice that by simply rearranging the terms of the denominator of the very first step, a solution becomes evident This occurs without additional involvement, thus lessening the chance of error

$$\frac{4a(6xz-9xy-10z+15y)}{2b(6xz-10z+15y-9xy)}=\frac{2a\cancel{(6xz-9xy-10z+15y)}}{b\cancel{(6xz-9xy-10z+15y)}}=\frac{2a}{b}$$

Another item of note

Although the following fractions are equivalent

$$\frac{-a}{b-c}=-\frac{a}{b-c}=\frac{a}{c-b}$$

the last expression is most preferable

EXERCISES 8-1

Replace the original fraction with an equivalent fraction that does not contain a minus sign in the numerator Simplify, where possible (Ex 1-6)

1. $\frac{-2x}{3-5x}$
2. $\frac{-a-b}{b-a}$
3. $-\frac{-3a^2-5ab}{5b-3a}$
4. $-\frac{-2x^2-6x}{x+3}$
5. $\frac{-x^2+9}{(x-3)^2}$
6. $\frac{-a^2+4b^2}{8b-4a}$

Reduce the following fractions to simplest terms

7. $\frac{-24a^2z}{-36az^2}$
8. $\frac{35b^2c}{-19b^2c-16b^2c}$
9. $\frac{3ab}{3ab-3a}$
10. $\frac{(3ab)^2}{9a^2b^2}$

11. $\dfrac{5xy^2}{5x + 15y}$

12. $\dfrac{15y - 15x}{-5xy^2}$

13. $\dfrac{a^2 - b^2}{a^2}$

14. $\dfrac{a^2b^2 - b^2}{a^2 - 1}$

15. $\dfrac{2x^2y - 8y}{4y - 4xy + x^2y}$

16. $\dfrac{(x - 3)^2}{x^3 - 27}$

17. $\dfrac{15ay - 18a + 3ay^2}{-9a^2 + 9a^2y^2}$

18. $\dfrac{15x^2 + x - 6}{3 - 17x + 20x^2}$

19. $\dfrac{(x + y)^2 - 4}{-6x + 3x^2 + 3xy}$

20. $\dfrac{(x - a)(x + b)}{(-b - x)(a - x)}$

21. $\dfrac{(2x - 3b) + (3x - 2b)}{(3b + 2x) - (2b + 3x)}$

22. $\dfrac{6x^2 - 9x - 6a + 4ax}{12ax + 18x^2 - 18a - 27x}$

23. $\dfrac{9a^2 - 30ab + 25b^2}{24a^2 - 10ab - 50b^2}$

24. $\dfrac{(x^2 - x - 6)(x - 1)}{(x^2 - 9)(x^2 - 4x + 3)}$

25. $\dfrac{a^2 - (b + c)^2}{(a + c)^2 - b^2}$

26. $\dfrac{(x - 3)(x + 3)}{(x - 3)^2 + x^2 - 9}$

27. $\dfrac{4a^2 - 4b^2 + 2(b - a)(b + a)(2)}{6a^2 - 12ab + 18b^2}$

28. $\dfrac{4a^2 - 4b^2}{(2a + b)^2 - [4a^2 + 4ab + b^2]}$

8-3 MULTIPLYING AND DIVIDING FRACTIONS

To find the **product** of algebraic fractions, **multiply** the corresponding **numerators** and **denominators**. The real numbers and literal numbers are combined according to procedures already established.

$$\frac{2x}{3y} \cdot \frac{5x}{7} = \frac{(2)(5)(x \cdot x)}{(3)(7)(y)} = \frac{10x^2}{21y}$$

Or in general form,

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd},$$

where the denominators are different from zero.

Before attempting to multiply or divide algebraic fractions, it is advisable to study the elements of the fraction for possible common multiples or factors. Common factors can be canceled out before the actual process of multiplication gets underway, thus eliminating some unnecessary detail. The attempt or goal in mathematics is always to work with simplified expressions within the control of the technician.

The guidelines involving multiplication apply also to division, since division is the inverse of multiplication. To **divide** two algebraic fractions,

invert the denominator (divisor) of the fraction and multiply it by the numerator (dividend) of the fraction

$$\frac{2a}{3b} \div \frac{4}{b} = \frac{2a}{3b} \times \frac{b}{4} = \frac{a}{6}$$

or

$$\frac{\frac{2a}{3b}}{\frac{4}{b}} = \frac{2a}{3b} \times \frac{b}{4} = \frac{a}{6}$$

In general form,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

The divisor in general form is c/d, when inverted, c/d becomes d/c These expressions are called *reciprocals*, that is, c/d is the reciprocal of d/c

In general form,

$\dfrac{1}{a}$ is the reciprocal of a

EXAMPLE 8 E

Multiply the following expression

$$\frac{2ax}{7by} \cdot \frac{21b}{6a} \cdot \frac{(2x - y)}{(y - 2x)}$$

Solution

Before carrying out the somewhat involved multiplication, a study for possible common multiples or factors should be made To facilitate this concept, the problem will be rewritten with preliminary grouping

$$\frac{(2)(21)(abx)(2x - y)}{(7)(6)(aby)(y - 2x)}$$

which leads further to

$$\frac{(2)(21)(abx)(2x - y)}{(7)(6)(aby)(y - 2x)} = \frac{x(2x - y)}{y(y - 2x)} \quad \text{(after canceling)}$$

Using the principles of signs,

$$y - 2x = -(2x - y),$$

and substituting accordingly leads to the completion of the problem

$$\frac{x(2x - y)}{y(y - 2x)} = \frac{x\cancel{(2x - y)}}{-y\cancel{(2x - y)}} = -\frac{x}{y}$$

This is identical to the procedure used previously for simplifying fractions.

EXAMPLE 8-F:

Simplify

$$\frac{2a}{x-3}\cdot\frac{2x^2-x-15}{3x-2}\div\frac{2ax+5a}{2-3x}.$$

Solution:

The initial step should involve studying all the elements of the given expression to determine if at least some of them can be factored. In this example, consideration is narrowed down to $2x^2 - x - 15$ and $2ax + 5a$.

Factoring:

$$2x^2 - x - 15 = (2x+5)(x-3)$$

and

$$2ax + 5a = a(2x+5)$$

Furthermore,

$$\frac{2a}{x-3}\cdot\frac{2x^2-x-15}{3x-2}\div\frac{2ax+5a}{2-3x}=$$

$$\frac{2a}{x-3}\cdot\frac{(2x+5)(x-3)}{3x-2}\div\frac{a(2x+5)}{2-3x}$$

Inverting the divisor and changing $(2-3x)$ to $-(3x-2)$ leads to completion of the problem.

$$\frac{2a}{\cancel{(x-3)}}\cdot\frac{\cancel{(2x+5)}\cancel{(x-3)}}{\cancel{(3x-2)}}\cdot\frac{-\cancel{(3x-2)}}{a\cancel{(2x+5)}}=\frac{-2a}{a}=-2$$

EXERCISES 8-2

Find the reciprocal of each of the given expressions (Ex. 1-6).

1. $\frac{2x}{3}$
2. $\frac{x}{3}$
3. $\frac{1}{3}$
4. $\frac{(a+b)}{(b-a)}$
5. $\frac{\frac{1}{2}(x-2y)}{3}$
6. $\frac{3x^2(a^2-b^2)}{5y^2}$

Perform the indicated operations and reduce to lowest terms.

7. $\frac{12x^2}{5y^2}\cdot\frac{15y}{4x}$
8. $\frac{12x^2}{5y^2}\div\frac{15y}{4x}$
9. $\frac{9ab^2}{7a^2c}\cdot\frac{49a^3c^3}{27b^3}$
10. $\frac{49x^2y}{8xy^3}\div\frac{21xy}{16x^2y^2}$

11. $\dfrac{3x-5}{4y+2}\cdot\dfrac{16y^2-4}{25-9x^2}$

12. $\dfrac{16y^2-4}{25-9x^2}-\dfrac{4y+2}{3x-5}$

13. $\dfrac{y^2+2y+1}{16a^2x}\cdot\dfrac{36ax^2}{y^2-1}$

14. $\dfrac{2y^2+3y+1}{72ay}-\dfrac{y^2+3y+2}{9a^2y^2}$

15. $\dfrac{(x-2)^2-16}{2x^2-9x+7}\cdot\dfrac{(x-1)^2}{x^2-4}-\dfrac{6-7x+x^2}{-3(2x-7)}$

16. $\dfrac{a+b}{a^2-b^2}\cdot\dfrac{(a-b)^2}{(a+b)^2}-\dfrac{a^2-b^2}{(a-b)^2}$

17. $\dfrac{a}{a^2-b^2}\cdot\dfrac{(a-b)^2}{3a^2}\cdot\dfrac{3a(a+b)}{(b-a)}-\dfrac{6a^2(a-b)}{3a(b-a)}$

18. $\dfrac{25y^2-25y-14}{25y^2-45y+14}\cdot\dfrac{2-5y}{14+25y-25y^2}-\dfrac{1}{42-30y}$

19. $\dfrac{9b^2-16a^2}{2a^2-7ab-4b^2}\cdot\dfrac{6a+14b}{6b+8a}\cdot\dfrac{4a^2-4ab-3b^2}{6a^2+5ab-21b^2}$

20. $\dfrac{10xy-2x-5y+1}{3xy-2+3x-2y}\cdot\dfrac{y^2+4y+3}{6xy+4x-2-3y}-\dfrac{5y^2+14y-3}{9xy-4-6y+6x}$

8-4 ADDING AND SUBTRACTING FRACTIONS

The procedure for finding the sum or difference of algebraic fractions follows the principle established for adding and subtracting arithmetic fractions

Several fractions can be added or subtracted only if they have a common denominator Once this has been established, the numerators of the several fractions can be combined algebraically The denominator of the new fraction will be the common denominator of the fractions involved

In general form,

$$\frac{a}{b}+\frac{c}{b}-\frac{d}{b}=\frac{a+c-d}{b}$$

It should be pointed out, however, that

$$\frac{b}{a+c-d}\neq\frac{b}{a}+\frac{b}{c}-\frac{b}{d}$$

By way of illustration,

$$\frac{3}{4}+\frac{5}{4}-\frac{7}{4}=\frac{3+5-7}{4}=\frac{8-7}{4}=\frac{1}{4}$$

On the other hand,

$$\frac{4}{3+5-7}\neq\frac{4}{3}+\frac{4}{5}-\frac{4}{7}$$

However,

$$\frac{4}{3+5-7}=\frac{4}{8-7}=\frac{4}{1}=4$$

From the brief discussion along with previous experience in fractions (Chap. 2), it appears that the initial step in the process of adding and subtracting fractions is to replace fractions with equivalent fractions to meet the criterion of *lowest common denominators.*

The *least or lowest common denominator* is an algebraic expression that contains the smallest number of common multiples or prime factors of all of the denominators of the given fractions (zero excluded).

EXAMPLE 8-G:

Combine the fractions, as indicated.

$$\frac{3x-2y}{9}+\frac{5x+3y}{12}-\frac{7x+y}{18}$$

Solution:

First, find the prime factors of all the denominators.

$$9=(3)(3),\ 12=(2)(2)(3),\ 18=(2)(3)(3)$$

The prime factors to consider here are 2 and 3. The largest number of times 2 appears in any given term is twice, which is also true of 3. Therefore, the lowest common multiple of the given denominators is $(2)(2)(3)(3)=36$, which is also called the *least or lowest common denominator.* Notice further that 36 is the smallest quantity into which all the given denominators will divide evenly.

$$\frac{36}{9}=4,\ \frac{36}{12}=3,\text{ and }\frac{36}{18}=2$$

Thus, the least, or lowest, common denominator can be viewed as the smallest quantity that contains each of the given denominators. Once the lowest common denominator has been established, the next step involves replacing original fractions with equivalent fractions containing the new denominator. This step is in accordance with the properties of fractions. Hence,

$$\frac{3x-2y}{9}=\frac{4(3x-2y)}{4(9)}=\frac{12x-8y}{36}$$

$$\frac{5x+3y}{12}=\frac{3(5x+3y)}{3(12)}=\frac{15x+9y}{36}$$

$$\frac{7x+y}{18}=\frac{2(7x+y)}{2(18)}=\frac{14x+2y}{36}$$

Thus,

$$\frac{3x-2y}{9}+\frac{5x+3y}{12}-\frac{7x+y}{18}=\frac{12x-8y}{36}+\frac{15x+9y}{36}-\frac{14x+2y}{36}$$

$$=\frac{12x-8y+15x+9y-14x-2y}{36}=\frac{13x-y}{36}$$

EXAMPLE 8 H

Combine the following fractions

$$\frac{2x}{2x-3}-\frac{x-2}{4x^2-9}-\frac{2x^2}{2x^2+x-6}$$

Solution

The initial step suggests finding the prime factors of the denominators $4x^2-9=(2x-3)(2x+3)$, $2x^2+x-6=(2x-3)(x+2)$, $(2x-3)$ appears in lowest terms

By inspection, the prime factors of the lowest common denominator are $(2x-3)(2x+3)(x+2)$, which means that each denominator is contained in this expression, hence the lowest common denominator

Next, convert the original fractions to equivalent fractions What really takes place now is that each fraction is multiplied (both numerator and denominator) by those factors not contained in the original denominators

$$\text{LCD}\quad (2x-3)(2x+3)(x+2)=(4x^2-9)(x+2)$$

Thus,

$$\frac{2x}{2x-3}=\frac{(2x+3)(x+2)(2x)}{(2x+3)(x+2)(2x-3)}=\frac{4x^3+14x^2+12x}{(4x^2-9)(x+2)}$$

$$\frac{x-2}{(2x-3)(2x+3)}=\frac{(x+2)(x-2)}{(x+2)(2x-3)(2x+3)}=\frac{x^2-4}{(4x^2-9)(x+2)}$$

$$\frac{2x^2}{(2x-3)(x+2)}=\frac{(2x+3)(2x^2)}{(2x+3)(2x-3)(x+2)}=\frac{4x^3+6x^2}{(4x^2-9)(x+2)}$$

Replacing original fractions with equivalent fractions leads to completion of the problem

$$\frac{4x^3+14x^2+12x}{(x+2)(4x^2-9)}-\frac{x^2-4}{(x+2)(4x^2-9)}-\frac{4x^3+6x^2}{(x+2)(4x^2-9)}$$

$$=\frac{4x^3+14x^2+12x-x^2+4-4x^3-6x^2}{(x+2)(4x^2-9)}=\frac{7x^2+12x+4}{(x+2)(4x^2-9)}$$

EXERCISES 8 3

Find the lowest common multiple for each set of expressions (Ex 1-8)

1. $2x, 3x^2, 24$

2. $a, a + b, 5a$

3. $3a^2, 6a, a^2 - 1$

4. $4acx^2, 12a^2cx, 16ac^2x$

5. $(5x - 5), (25x^2 - 25), (x + 1)$

6. $(ax^2 + bc + bx + acx), (ax^2 + bx - acx - bc)$

7. $(9x^2 + 3x - 2), (3x^2 - 8x - 3), (9x^2 - 1)$

8. $(9x^2 + 12x + 4), (4 - 12x^2 + 9x), (4 - 9x^2)$

Combine (simplify) the following fractions as indicated.

9. $\dfrac{3}{ax} - \dfrac{2}{bx} + \dfrac{6}{cx}$

10. $\dfrac{3a}{2x} + \dfrac{3a}{x - 1}$

11. $\dfrac{16}{x - 4} - \dfrac{12}{x - 2} + \dfrac{3}{4 - x}$

12. $\dfrac{1}{D_1} + \dfrac{1}{D_2} - \dfrac{1}{f}$

13. $\dfrac{3x}{4x^2y} - \dfrac{2y}{8xy^2} - \dfrac{4xy}{16x^2y^2}$

14. $\dfrac{2y}{5 + 2x} - \dfrac{3y}{2x - 5}$

15. $\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$

16. $\dfrac{E}{R} - \dfrac{e}{R}$

17. $\dfrac{6}{2 - 3x} + \dfrac{9x(1 + 2x)}{9x^2 - 4}$

18. $\dfrac{1}{x} - \dfrac{3}{2(x + 2)} - \dfrac{3}{(x + 2)}$

19. $\dfrac{6b}{20a^2 + ab - 12b^2} + \dfrac{5a}{6b^2 + ab - 12a^2}$

20. $\dfrac{4x}{x^2 + 3x + 2} + \dfrac{2}{x + 1} - \dfrac{5}{x + 2}$

21. $\dfrac{3x + 2}{6x} - (5x - 3)$

22. $\dfrac{3x^2 - 5x + 2}{x - 1} - \dfrac{5x^2 + 8x + 3}{x + 1}$

23. $\dfrac{16y^2 - 9}{3 - 4y} - \dfrac{25 - 9y^2}{3y - 5}$

24. $\dfrac{1}{4x^2 - 16} - \dfrac{1}{x - 2} - \dfrac{1}{4} + 1$

25. $\dfrac{x - 1}{12x^2 - 13x + 3} + \dfrac{x - 1}{4x^2 - 11x + 6} - \dfrac{x - 1}{3x^2 - 7x + 2}$

An expression that contains fractions in the numerator and/or denominator is called a *complex fraction* In general form,

$$\frac{\frac{a}{b}-c}{\frac{a}{b}+d} \text{ represents a complex fraction}$$

A complex fraction may be simplified by first reducing the fraction(s) in the numerator and then reducing the fraction(s) in the denominator, followed by the indicated division It is important to keep the *main line of division intact* Numerator and denominator must be clearly separated until they are combined through division

It might be advisable to review section 2-6 before proceeding with this topic

EXAMPLE 8-I

Simplify

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}$$

Solution

Combining fractions in numerator and denominator reduces the complex fraction to a more meaningful expression

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}=\frac{\frac{y+x}{xy}}{\frac{y-x}{xy}}$$

Inverting denominator and multiplying·

$$\frac{y+x}{\cancel{xy}}\cdot\frac{\cancel{xy}}{y-x}=\frac{y+x}{y-x},$$

which is in final form

EXAMPLE 8-J:

Simplify

$$\frac{\frac{2}{x+1}-\frac{3}{x-1}}{\frac{1}{x}-\frac{x}{x^2-1}}$$

Solution·

First, combine fractions in the numerator and the denominator

$$\frac{\dfrac{2}{x+1}-\dfrac{3}{x-1}}{\dfrac{1}{x}-\dfrac{x}{x^2-1}} = \frac{\dfrac{2(x-1)}{(x-1)(x+1)}-\dfrac{3(x+1)}{(x+1)(x-1)}}{\dfrac{(x^2-1)}{x(x^2-1)}-\dfrac{(x)(x)}{x(x^2-1)}}$$

$$= \frac{\dfrac{2x-2-3x-3}{x^2-1}}{\dfrac{x^2-1-x^2}{x(x^2-1)}} = \frac{\dfrac{-x-5}{x^2-1}}{\dfrac{-1}{x(x^2-1)}}$$

Next, invert and multiply.

$$\frac{-(x+5)}{\cancel{(x^2-1)}} \cdot \frac{x\cancel{(x^2-1)}}{-1} = \frac{-x(x+5)}{-1} = x(x+5)$$

EXAMPLE 8-K:

Simplify

$$\frac{a-\dfrac{2}{a-1}}{\dfrac{a-\dfrac{2}{a-1}}{a-\dfrac{1}{a}}}$$

Solution:

Combine both numerator and denominator accordingly.

$$\frac{a-\dfrac{2}{a-1}}{\dfrac{a-\dfrac{2}{a-1}}{a-\dfrac{1}{a}}} = \frac{\dfrac{a^2-a-2}{a-1}}{\dfrac{\dfrac{a^2-a-2}{a-1}}{\dfrac{a^2-1}{a}}} = \frac{\dfrac{(a-2)(a+1)}{a-1}}{\dfrac{(a-2)\cancel{(a+1)}}{a-1} \cdot \dfrac{a}{(a-1)\cancel{(a+1)}}}$$

$$= \frac{\dfrac{(a-2)(a+1)}{(a-1)}}{\dfrac{a(a-2)}{(a-1)(a-1)}}$$

Invert and multiply.

$$\frac{\cancel{(a-2)}(a+1)}{\cancel{(a-1)}} \cdot \frac{\cancel{(a-1)}(a-1)}{a\cancel{(a-2)}} = \frac{a^2-1}{a}$$

EXERCISES 8-4

Simplify.

1. $\dfrac{1-\dfrac{1}{a}}{1+\dfrac{1}{a}}$

2. $\dfrac{a-\dfrac{2}{a^2}}{1+\dfrac{2}{a}}$

3. $\dfrac{\frac{1}{a}-\frac{1}{\frac{1}{a}}}{a-\frac{1}{\frac{1}{a}}}$

4. $\dfrac{2+\frac{2}{(2x-1)}}{1-\frac{2x}{2x+1}}$

5. $\dfrac{1+\frac{1}{R_1}}{1+\frac{1}{R_2}+\frac{1}{R_3}}$

6. $\dfrac{1}{\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}}$

7. $\dfrac{\frac{1}{x-1}-\frac{x}{x^2-1}}{\frac{1}{x-1}+\frac{1}{x+1}}$

8. $\dfrac{\frac{3b}{2a}-\frac{1}{\frac{2a}{3b}}}{3a-\frac{1}{\frac{2a}{3b}}+2b}$

9. $\dfrac{5x}{2y-1}-\dfrac{\frac{5x}{2y-1}}{2y+1}+3x$

10. $\dfrac{1}{\frac{1}{a+b}}-b-\dfrac{1}{\frac{1}{a+b}}$

11. $\dfrac{\frac{ax}{ax-1}+\frac{2a}{\frac{1}{ax+1}}}{\frac{ax-1}{ax}+\frac{\frac{1}{ax+1}}{2a}}$

12. $\dfrac{R_e}{1-\frac{R}{R+nR_e}}$

13. $\dfrac{R}{R+1}+\dfrac{3R}{R+2}-5$

14. $\dfrac{\frac{1}{r_1}}{r_1+r_2}+\dfrac{\frac{1}{r_2}}{r_2+r_3}$

15. $\dfrac{E}{R+\frac{r}{n}}$

REVIEW EXERCISES 8-5

Simplify the following algebraic expressions

1. $\dfrac{2a(x-2)}{(x+2)}\cdot\dfrac{(x^2-4)}{4a^2}-\dfrac{(x-2)^2}{2a}$

2. $\dfrac{(x-1)}{(x^2-36)} \cdot \dfrac{(6-x)}{(x^2+1)} \div \dfrac{(1-x)}{(x+1)^2(6+x)}$

3. $\dfrac{\dfrac{ab+1}{ab-1}+1}{1-\dfrac{ab+1}{ab-1}}$

4. $\dfrac{\dfrac{24ay^2}{15a^2y}}{\dfrac{6a^2y}{25ay^2}}$

5. $\dfrac{15ay-6a^2-35y^2+14ay}{4a^2-25y^2}$

6. $\dfrac{16b^2x}{5bx^2} \div \dfrac{64b^3x^2}{35b^2x^3}$

7. $3+\dfrac{a}{5a-1}+\dfrac{4a}{1-5a}$

8. $\dfrac{(3x-3y)}{9} \cdot \dfrac{(y+x)^2}{(x^2-y^2)} \cdot \dfrac{(x+y)}{(x^2+y^2)}$

9. $\dfrac{y^2+2y-3}{3y^2-2y-1} \div \dfrac{y^2+6y+9}{9y^2+6y+1}$

10. $\dfrac{(x^3-y^3)}{(x^2-2xy+y^2)} \cdot \dfrac{(x^2-y^2)}{(x+y)} \div \dfrac{(x^2+xy+y^2)}{(x-y)^3}$

11. $\dfrac{2}{3a^2}-\dfrac{5}{ab}+\dfrac{3}{b^2}$

12. $\dfrac{x}{25-9y^2}+\dfrac{y}{3y-5}$

13. $\dfrac{2}{x^2+12x+36}-\dfrac{3}{x^2-12x+36}+\dfrac{1}{x^2-36}$

14. $\dfrac{5}{a}-\dfrac{2}{1-3a}+\dfrac{a+1}{3a+1}$

15. $\dfrac{1-\dfrac{1}{x-a}}{a-\dfrac{a}{x-a}}$

16. $\dfrac{1-\dfrac{\frac{1}{x}}{\frac{3}{4x}}}{x-\dfrac{\frac{3}{4x}}{\frac{1}{2}}}$

17. $\dfrac{2R}{R_1 + R_2} - \dfrac{R}{R_2 - R_1}$

18. $\dfrac{(a-2) - \dfrac{2a}{a+2}}{a + 2 + \dfrac{2a}{a-2}}$

19. $\dfrac{x^3 - 27y^3}{x^2 - 6xy + 9y^2} \cdot \dfrac{x^2 - 9y^2}{x^2 - 3xy + 9y^2} - \dfrac{x^2 + 3xy + 9y^2}{x^3 + 27y^3}$

20. $1 - \dfrac{\dfrac{1}{a}}{1 + \dfrac{1}{a}} - a - \dfrac{1 + \dfrac{1}{a}}{\dfrac{\dfrac{1}{a}}{1 + \dfrac{1}{a}}}$

21. Find the current, I, for a particular parallel grouping of cells if $I = E/(Re + n/Ri)$, $E = 7.5$ v, $n = 5$, $Ri = 0.5$ ohm, and $Re = 750$ ohms

22. The joint resistance, R, of several elements in parallel is given by the formula $1/R = 1/R_1 + 1/R_2 + 1/R_3$. Find the joint resistance, R, if $R_1 = 10$ ohms, $R_2 = 25$ ohms, and $R_3 = 35$ ohms

CHAPTER 9

Exponents, Roots, and Radicals

Exponents and radicals are associated with an important concept of mathematics called **roots**. Exponents were discussed in an earlier chapter. At that time it was established that an exponent is a number that indicates how many times a factor or base appears in the product.

$$a^2 = (a)(a); \quad x^3 = (x)(x)(x); \quad 6^4 = (6)(6)(6)(6);$$
$$(3x^2 - y)^2 = (3x^2 - y)(3x^2 - y)$$

9-1 ROOTS AND RADICALS

The radical, $\sqrt{}$, is a symbol defining a mathematical process that can be interpreted as the inverse of the principle just demonstrated. Briefly, an exponent indicates the number of times the base appears as a factor: $a^3 = (a)(a)(a)$. The radical, $\sqrt[3]{a^3}$, is associated with the process that defines the base or factor. The *base* or *factor* is *called the root* of the given expression.

$$\sqrt[3]{a^3} = \sqrt[3]{(a)(a)(a)} = a,$$

where a is called the root. *The number outside the radical is called the index of the root whereas the expression inside the radical is called the radicand.* The **index** reflects the number of equal factors that are needed to form a product equal to the radicand. The common factor or **base** is called the **root**. In general form:

$$\sqrt[n]{b} = a$$

***n* is the index of the root, *b* is the radicand,** and ***a* is the root.**

The same expression can also be expressed in exponential form.

Radical Form	Exponential Form
$\sqrt[n]{b} = a$	$a^n = b$

where *n ıs a rational number different from zero*

$$\sqrt[2]{36} = \pm 6, \qquad (\pm 6)^2 = 36$$

Square root carries an ındex of 2 but ıt ıs seldom ındıcated

$$\sqrt[2]{36} = \sqrt{36} = \pm 6$$

Cube root ıs associated with an ındex of 3, and so on

Extracting roots can become a complex operation ıf the ındex ıs other than a multiple of 2 or 3

$$\sqrt[2\,5]{0\,531}, \qquad \sqrt[3\,2]{14\,06},$$

By the definition of a square root, $\sqrt{36} = +6$ and -6, where $(+6)^2 = (-6)^2 = 36$

Representing a square root by a negative number may lead to an ambiguous statement For example $-\sqrt{36} = -(-6) = +6$, which ıs not acceptable, $-\sqrt{36} \neq -(-6)$ or $+6$

This condition ıs clarified by the ıntroduction of a concept called principal roots

9 2 PRINCIPAL ROOTS

$$\sqrt[n]{b} = a$$

Whenever the ındex of the root ıs even, there will be two roots whose absolute values are equal

$$\sqrt{81} = \pm 9, \qquad \sqrt[4]{16} = \pm 2$$

The principal root, when n ıs even, ıs defined as the positive root Thus,

$$\sqrt{81} = +9, \qquad \sqrt[4]{16} = +2,$$

Whenever the ındex of the root ıs odd and the radıcand ıs positive, the principal root ıs defined as positive

$$\sqrt[3]{27} = +3, \qquad \sqrt[3]{125} = +5$$

If the ındex ıs odd and the radıcand ıs negative, the principal root will be negative

$$\sqrt[3]{-27} = -3, \qquad \sqrt[3]{-125} = -5$$

When n ıs even and $b < 0$ (negative) the roots are called ımaginary and are defined ın an unique way

In general, the principal root will carry the same sign as the radıcand

EXAMPLE 9-A:

Find the principal roots of the various expressions.

1. $\sqrt{4} = 2$, $\sqrt[3]{8} = 2$, $\sqrt[3]{-8} = -2$
2. $\sqrt[4]{83{,}521} = 17$, $\sqrt[3]{-343} = -7$
3. $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $\sqrt{64x^2} = 8x$
4. $\sqrt[3]{-27y^6} = -3y^2$, $\sqrt[7]{y^{14}} = y^2$

A minus sign preceding a radical indicates that the root is to be multiplied by (-1).

$$-\sqrt[n]{b} = -(a) = -a$$
$$-\sqrt[3]{-125} = -(-5) = 5$$
$$-\sqrt[4]{256} = -(4) = -4$$
$$-\sqrt[5]{-32} = -(-2) = 2$$
$$-\sqrt{121} = -(11) = -11$$

Note: when n is even, $-\sqrt[n]{b} \neq \sqrt[n]{-b}$. The latter expression involves imaginary roots that will be defined later. $-\sqrt{100} \neq \sqrt{-100}$

EXERCISES 9-1

Express the written statements in radical form (Ex. 1-6).

1. cube root of $-\frac{216}{1{,}000}$
2. fourth root of 1,296
3. square root of 169
4. seventh root of -128
5. square root of $\frac{1}{36}$
6. sixth root of 10^6

Find the principal root of each of the following:

7. $\sqrt[3]{1{,}000}$
8. $\sqrt[3]{-1{,}000}$
9. $\sqrt[2]{676}$
10. $\sqrt[4]{6{,}561}$
11. $\sqrt[5]{-243}$
12. $\sqrt[3]{-8m^3n^3}$
13. $\sqrt{(x-2)^2}$
14. $\sqrt[3]{(x-y)^3}$
15. $\sqrt[3]{(x^3-y^3)^3}$
16. $\sqrt{(x^2-y^2)^4}$
17. $\sqrt[3]{-0.001}$
18. $\sqrt[3]{\frac{216}{1{,}728}}$

19. $\sqrt{\dfrac{10^3}{1000}}$

Find the values of the following

20. $\sqrt[3]{-27}$

21. $-\sqrt[3]{27}$

22. $-\sqrt[3]{-27}$

23. $-\sqrt[4]{256}$

24. $-\sqrt[3]{-8}+\sqrt[3]{-8}$

25. $-\sqrt{441}+\sqrt{441}$

26. $\sqrt[3]{-27}+\sqrt{9}-\sqrt{9}$

27. $-\sqrt[5]{-32}+\sqrt[5]{-32}$

28. $\sqrt{4x^2}-\sqrt{4x^2}$

9-3 FRACTIONAL EXPONENTS

Before discussing fractional exponents, the laws of exponents will be reviewed (Sec 4-2, 7-8)

$$a^0 = 1$$

$$a^m \cdot a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n} = \frac{1}{a^{n-m}}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

where m and n are positive integers Negative exponents can be expressed as positive exponents within this definition

$$a^{-n} = \frac{1}{a^n}$$

Very often the laws or principles of mathematics involving positive integers also apply to other positive rational numbers For example, the laws of exponents can be expanded to include positive fractional (rational) exponents. Thus,

$$a^{1/2} \cdot a^{1/2} = a^{1/2+1/2} = a^{2/2} = a$$

$$a^{1/3} \cdot a^{1/3} \cdot a^{1/3} = a^{1/3+1/3+1/3} = a^{3/3} = a$$

$$a^{2/3} \cdot a^{4/3} = a^{2/3+4/3} = a^{6/3} = a^2$$

$$(a^{1/2})^2 = a^{1/2(2)} = a$$

In general form,

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

where m and n are both positive integers and the fraction m/n appears in lowest terms

$$a^{5/3} = \sqrt[3]{a^5} = (\sqrt[3]{a})^5$$

The denominator of the fractional exponent, n, defines the **root** and the numerator, m, defines the power of the **base**.

If $m = n$,

$$\sqrt[n]{a^m} = \sqrt[n]{a^n} = a^{n/n} = a^1 = a$$

$$\sqrt[5]{b^5} = b, \sqrt[4]{x^4} = x, \sqrt[3]{8} = \sqrt[3]{2^3} = 2$$

Furthermore, if $m = 1$,

$$a^{m/n} = a^{1/n} = \sqrt[n]{a}$$

which defines the **fractional exponential form** in terms of the **radical form.**

$$a^{1/2} = \sqrt{a}, a^{1/3} = \sqrt[3]{a}$$

Finally,

$$a^{-m/n} = \frac{1}{a^{m/n}}, n \neq 0$$

A common error occurs because of the failure to distinguish between expressions such as $(2a)^{-1}$ and $2a^{-1}$.

$$(2a)^{-1} = \frac{1}{2a} \qquad 2a^{-1} = 2 \times \frac{1}{a} = \frac{2}{a}$$

$$(2a)^{-1} \neq 2a^{-1}$$

This distinction should be studied.

EXAMPLE 9-B:

Find the numerical value of $(64)^{2/3}$.

Solution:

In general form,

$$a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Thus,

$$(64)^{2/3} = \sqrt[3]{(64)^2} = (\sqrt[3]{64})^2 = (4)^2 = 16$$

Alternate Solution:

$$(64)^{2/3} = \sqrt[3]{(64)^2} = \sqrt[3]{4{,}096} = 16$$

EXAMPLE 9-C:

Find the numerical value of $(64)^{-2/3}$.

Solution:

$$(64)^{-2/3} = \frac{1}{(64)^{2/3}} = \frac{1}{16}$$

EXAMPLE 9-D.

Simplify $(a^{3/4})^4$

Solution:

$$(a^{3/4})^4 = a^{(3/4)(4)} = a^3$$

or

$$(a^{3/4})^4 = a^{3/4} \cdot a^{3/4} \cdot a^{3/4} \cdot a^{3/4} = a^{3/4+3/4+3/4+3/4} = a^{12/4} = a^3$$

EXERCISES 9-2

Express in radical form (Ex 1-6)

1. $17^{1/2}$ *2.* $(-8)^{1/3}$ *3.* $(-8)^{2/3}$

4. $(-5x)^{3/5}$ *5.* $(a^2)^{-2/3}$ *6.* $(4a^{-2})^{-3/2}$

Express in fractional exponential form (Ex 7-12)

7. $\sqrt[2]{a^3}$ *8.* $\sqrt[3]{a^2}$ *9.* $\sqrt[3]{xy}$

10. $\sqrt[5]{(2a)^3}$ *11.* $\sqrt[3]{a^2b^2}$ *12.* $-\sqrt[3]{-x^3y^6}$

Simplify the following expressions (Ex 13-25)

13. $(16)^{3/2}$ *14.* $-4(-8)^{2/3}$

15. $(169a^2b^4)^{3/2}$ *16.* $2x + (4x^2)^{1/2} - (-27x^3)^{1/3}$

17. $\left(\dfrac{125a^4b^3}{5a^2b}\right)^{1/2}$ *18.* $(81a^4)^{1/2} - (-27a^6)^{1/3}$

19. $(81a^4)^{-1/2} - (127a^6)^{-1/3}$ *20.* $(-216a^{-3}b^6)^{2/3}$

21. $(x^{1/2} - y^{1/2})(x^{1/2} + y^{1/2})$ *22.* $(x^{1/2} - y^{1/2})^2$

23. $(x^{1/3} - y^{1/3})^3$ *24.* $(y^{1/3} + 1)(y^{2/3} - y^{1/3} + 1)$

25. $[(4x^2 - 1)^{3/2}]^{2/3}$

Simplify the following The final form should not contain any negative exponents

26. $(ab)^{-2}$ *27.* $(3a^{-2})(3a^2)$

28. $\dfrac{a^{-2}b^3}{c^{-1}}$ *29.* $(x^{-3}) - (x^3)$

30. $(x^3) - (x^{-3})$ *31.* $x^{-2} + y^{-2}$

32. $(4a^{-2})^{-3/2}$ *33.* $x^{-1} + \dfrac{1}{x}$

34. $(a^{-2} - b^{-2})^{-2}$ *35.* $(y^{-5})(y^3)(y^{-6}) - (y^{-7})^2$

36. $(a^{-2} - b^{-2})^0(a^{-1} - b^{-2})^{-1}$

It becomes necessary at times to change the form of a radical as a matter of convenience to minimize arithmetic computations. The following laws of radicals, based primarily on the laws of exponents, often lead to the simplification of a given expression.

$$\sqrt[n]{a^n} = (\sqrt[n]{a})^n = a$$
$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$
$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

where n is positive and a and b are other than zero.

EXAMPLE 9-E:

By removing factors, simplify $\sqrt{48}$.

Solution:

An expression inside a radical can be treated as any other monomial (or polynomial) in terms of algebraic operations. Caution must be exercised, however, when attempting to remove a factor.

From the law of radicals,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

Thus,

$$\sqrt{48} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$$

The distinction between,

$$\sqrt{32 + 16} = \sqrt{48} \text{ and } \sqrt{16 \cdot 3} = \sqrt{48}$$

should be observed.

$$\sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3}$$

However,

$$\sqrt{32 + 16} \neq \sqrt{36} + \sqrt{16}$$

Note: The law involves **factors** not **sums.**

EXAMPLE 9-F:

Simplify $\sqrt[3]{648x^4y^5}$.

Solution:

Factor and remove any factor that is a perfect cube.

$$\sqrt[3]{648x^4y^5} = (\sqrt[3]{216} \cdot \sqrt[3]{3})(\sqrt[3]{x^3}\ \sqrt[3]{x})(\sqrt[3]{y^3} \cdot \sqrt[3]{y^2})$$
$$= (6\sqrt[3]{3})(x\sqrt[3]{x})(y\sqrt[3]{y^2}) = 6xy\sqrt[3]{3xy^2}$$

(simplified form)

Alternate Solution

Factor original expression within the radical and remove perfect cubes.

$$\sqrt[3]{648x^4y^5} = \sqrt[3]{(216)(3)(x^3)(x)(y^3)(y^2)} = 6xy\sqrt[3]{3xy^2}$$

One of the more undesirable and time consuming arithmetic computations involves division by an irrational number, such as $3/\sqrt{2}$, $\sqrt{5}/\sqrt[3]{7}$. The problem can be simplified by eliminating the radical in the denominator by a process referred to as rationalizing the denominator. *Rationalizing involves multiplying the numerator and denominator of the fraction by a factor that will convert the denominator into a perfect nth power.*

$$\sqrt[n]{\frac{a}{b}} = \sqrt[n]{\frac{a}{b} \cdot \frac{c}{c}}$$

where $(c) \times (b) = b^n$

$$= \sqrt[n]{\frac{ac}{b^n}} = \frac{\sqrt[n]{ac}}{b} \qquad (c \neq 0)$$

EXAMPLE 9-G

Rationalize the fraction $\sqrt[3]{16/9}$ and simplify.

Solution

The denominator can be rationalized by multiplying the numerator and denominator by a factor such that the new denominator is a perfect power of the index of the radical.

In the given example, the factor must yield a product that is a perfect cube, preferably the smallest cube larger than 9, which turns out to be 27. Thus,

$$\sqrt[3]{\frac{16}{9}} = \sqrt[3]{\frac{16}{9} \cdot \frac{3}{3}} = \sqrt[3]{\frac{48}{27}} = \frac{\sqrt[3]{48}}{3}$$

At this point the fraction is rationalized but not simplified, since $48 = 6 \times 8$ and 8 is a perfect cube.

Therefore,

$$\sqrt[3]{\frac{16}{9}} = \sqrt[3]{\frac{16}{9} \cdot \frac{3}{3}} = \sqrt[3]{\frac{8 \cdot 6}{27}} = \frac{2\sqrt[3]{6}}{3}$$

EXAMPLE 9-H

Simplify

$$\sqrt{\frac{7a}{6x^3y}}$$

Solution:

$$\sqrt{\frac{7a}{6x^3y}} = \sqrt{\frac{7a}{6x^3y}\cdot\frac{6xy}{6xy}} = \sqrt{\frac{42axy}{36x^4y^2}} = \frac{\sqrt{42axy}}{6x^2y}$$

EXAMPLE 9-I:

Rationalize the expression

$$\sqrt{\frac{x(x-y)}{x+y}}.$$

Solution:

Multiply the numerator and denominator by the factor $(x+y)$ and remove any perfect squares.

$$\sqrt{\frac{x(x-y)}{x+y}} = \sqrt{\frac{x(x-y)(x+y)}{(x+y)(x+y)}} = \sqrt{\frac{x(x^2-y^2)}{(x+y)^2}} = \frac{\sqrt{x(x^2-y^2)}}{x+y}$$

EXERCISES 9-3

Simplify by removing appropriate factors.

1. $\sqrt{72}$
2. $\sqrt[3]{2000}$
3. $\sqrt{24x^3}$
4. $\sqrt[3]{24x^3}$
5. $-\sqrt{a^3b^5}$
6. $\sqrt[3]{-(a^3b^5)}$
7. $\sqrt{200x^3y}$
8. $\sqrt[3]{-250a^9b^6}$
9. $\sqrt{\frac{a^4b}{x^2y^4}}$
10. $-2xy\sqrt[3]{\frac{-192x^4}{y^2}}$
11. $\sqrt{\frac{25x^2}{9y^2}}$
12. $\sqrt{\frac{45y^4}{196x^2}}$
13. $\sqrt{\frac{2xy}{7}}$
14. $\sqrt[3]{\frac{4ab}{25x^2}}$
15. $\sqrt{\frac{a+b}{3(a-b)}}$
16. $\sqrt{1+\frac{1}{a}}$
17. $\sqrt[3]{\frac{x+y}{72(x^2-y^2)}}$
18. $\sqrt[3]{\frac{x+y}{x^2-2xy+y^2}}$
19. $\sqrt{R-\frac{r}{R}}$
20. $\sqrt{a+2+\frac{1}{a}}$

9-5 ADDING AND SUBTRACTING RADICALS

Expressions containing radicals may be combined, subject to the principles established for the basic operations of other algebraic expressions. *With respect to addition and subtraction, like terms are combined and unlike terms are left as an indicated sum. Similar radicals are defined as those having the same index and radicand.*

$$2\sqrt[3]{5} + 3\sqrt[3]{5} - \sqrt[3]{5} = 4\sqrt[3]{5}$$
$$3\sqrt{a} - 2\sqrt{a} + 4\sqrt{b} = \sqrt{a} + 4\sqrt{b}$$

The radicands should be reduced to simplest form before proceeding with the addition or subtraction

EXAMPLE 9 J

Simplify

$$3\sqrt{2} - \sqrt{8} + 2\sqrt{98}$$

Solution

Study each term for possible factors and reduce accordingly Hence

$$\sqrt{8} = \sqrt{4 \cdot 2} = 2\sqrt{2}$$
$$2\sqrt{98} = 2\sqrt{49 \cdot 2} = 2 \cdot 7\sqrt{2} = 14\sqrt{2}$$

Therefore,

$$3\sqrt{2} - \sqrt{8} + 2\sqrt{98} = 3\sqrt{2} - 2\sqrt{2} + 14\sqrt{2} = 15\sqrt{2}$$

EXAMPLE 9 K

Combine terms as indicated

$$\sqrt[3]{\frac{x^2}{32}} - \frac{\sqrt[3]{16x^2}}{8} + \frac{3\sqrt[3]{2x^2}}{\sqrt[3]{64}}$$

Solution

First rationalize the denominators and simplify the numerators

$$\sqrt[3]{\frac{x^2}{32}} - \frac{\sqrt[3]{16x^2}}{8} + \frac{3\sqrt[3]{2x^2}}{\sqrt[3]{64}} = \sqrt[3]{\frac{x^2}{32} \cdot \frac{2}{2}} - \frac{\sqrt[3]{8 \cdot 2x^2}}{8} + \frac{3\sqrt[3]{2x^2}}{4}$$
$$= \sqrt[3]{\frac{2x^2}{64}} - 2\frac{\sqrt[3]{2x^2}}{8} + 3\frac{\sqrt[3]{2x^2}}{4} = \frac{\sqrt[3]{2x^2}}{4} - \frac{\sqrt[3]{2x^2}}{4} + \frac{3\sqrt[3]{2x^2}}{4}$$

Since the fractions have similar radicals, and a common denominator, they can be combined according to the principles established for other algebraic fractions Thus,

$$\frac{\sqrt[3]{2x^2}}{4} - \frac{\sqrt[3]{2x^2}}{4} + 3\frac{\sqrt[3]{2x^2}}{4} = 3\frac{\sqrt[3]{2x^2}}{4}$$

EXERCISES 9 4

Perform the indicated operations and simplify

1. $2\sqrt{3} - 4\sqrt{3} + 3\sqrt{3}$
2. $3a\sqrt[3]{b} + 2a\sqrt[3]{b} - a\sqrt[3]{b}$
3. $3\sqrt{27} - 2\sqrt{12} + 4\sqrt{108}$
4. $4\sqrt[3]{128} - 3\sqrt{250} + \sqrt{2{,}000}$

5. $5\sqrt{3} - 2\sqrt{\frac{1}{3}} + 3\sqrt{\frac{1}{147}}$

6. $8\sqrt{a^3} - 6a\sqrt{a} + \frac{3\sqrt{a^5}}{a}$

7. $6\sqrt[3]{4a^5b^4} - \frac{2a}{b}\sqrt[3]{32a^2b^7} + \frac{3b}{a}\sqrt[3]{\frac{a^8b}{16}}$

8. $5\sqrt{(a^2 - b^2)} + \sqrt{\frac{(a+b)}{(a-b)}}$

9. $12a\sqrt[3]{3b^3} - \frac{4}{b}\sqrt[3]{192a^3b^6} + ab\sqrt[3]{\frac{1}{25}}$

10. $\sqrt{a^2} - 2\sqrt{a^3} + 5\sqrt{a}$

11. $2\sqrt{(a^2-1)^3} - \sqrt{(a^2-1)}$

12. $4x\sqrt[3]{(a+y)^4} - 2x\sqrt[3]{a+y}$

13. $\sqrt{16x^2 + 32xy + 16y^2} + \sqrt{25x^2 + 50xy + 25y^2}$

14. $\sqrt{\frac{2x-1}{2x+1}} + \sqrt{\frac{2x+1}{2x-1}}$

15. $\frac{\sqrt{3}x^2 - \sqrt{12x^4}}{\sqrt{3x}}$

9-6 MULTIPLYING AND DIVIDING RADICALS

The principle of multiplying and dividing radicals is set forth in the laws of radicals.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

and

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Radicals of the **same order** can be **multiplied** and **divided** under **guidelines** established for other **algebraic expressions.**

EXAMPLE 9-L:

Multiply the given expressions and simplify the product.

$$3\sqrt{x^2y} \cdot 2\sqrt{xy^2}$$

Solution:

Combine the terms outside the radicals in the usual manner and continue with the terms inside the radicals, since they are of the same order.

$$3\sqrt{x^2y} \cdot 2\sqrt{xy^2} = 2 \cdot 3\sqrt{(x^2y)(xy^2)} = 6\sqrt{x^3y^3} = 6xy\sqrt{xy}$$

When the indices are identical, the radicals perform the same function as other symbols of grouping.

Alternate Solution:

Simplify first and then multiply.

$$3\sqrt{x^2y} \cdot 2\sqrt{xy^2} = 3x\sqrt{y} \cdot 2y\sqrt{x} = 6xy\sqrt{xy}$$

Note A product of the form

$$3\sqrt{x^2y}\cdot 2\sqrt[3]{xy^2} = (6x\sqrt{y})(\sqrt[3]{xy^2})$$

cannot be combined beyond the indicated product

EXAMPLE 9 M

Find the quotient of

$$\frac{5\sqrt[3]{x^5}}{\sqrt[3]{5x}}$$

Solution

Since both parts of the fraction are of the same order, they can be combined under one radical and simplified

$$\frac{5\sqrt[3]{x^5}}{\sqrt[3]{5x}} = 5\sqrt[3]{\frac{x^5}{5x}} = 5\sqrt[3]{\frac{x^4}{5}} = 5x\sqrt[3]{\frac{x}{5}}$$

Rationalize the denominator by multiplying, appropriately, by 25

$$5x\sqrt[3]{\frac{x}{5}} = 5x\sqrt[3]{\frac{x}{5}\cdot\frac{25}{25}} = 5x\sqrt[3]{\frac{25x}{125}} = \frac{5x}{5}\sqrt[3]{25x} = x\sqrt[3]{25x}$$

Many expressions of the form $4/(\sqrt{2} - x)$ $a/(\sqrt{b} - c)$, appear in the field of mathematics and technology *The denominator of these expressions is referred to as an irrational binomial* In order to simplify a fraction of this form, the denominator must be **rationalized** This can be accomplished by multiplying both numerator and denominator by the **conjugate** of the denominator

The conjugate differs from the original expression only by a change in sign of the second term

The conjugate of $\sqrt{2} - x$ is $\sqrt{2} + x$
The conjugate of $-\sqrt{3a} + b$ is $-\sqrt{3a} - b$
Or the conjugate of $b - \sqrt{3a}$ is $b + \sqrt{3a}$

EXAMPLE 9 N

Find the quotient and simplify

$$\frac{2\sqrt{3} + 3\sqrt{2}}{3\sqrt{2} - 2\sqrt{3}}$$

Solution

Rationalize the denominator by multiplying both parts of the fraction by the conjugate of $(3\sqrt{2} - 2\sqrt{3})$, which is $(3\sqrt{2} + 2\sqrt{3})$

$$\frac{(2\sqrt{3}+3\sqrt{2})}{(3\sqrt{2}-2\sqrt{3})}\frac{(3\sqrt{2}+2\sqrt{3})}{(3\sqrt{2}+2\sqrt{3})}=\frac{6\sqrt{6}+(9)(2)+(4)(3)+6\sqrt{6}}{(9)(2)-(4)(3)}$$
$$=\frac{30+12\sqrt{6}}{6}=5+2\sqrt{6}$$

Notice that the concept of conjugates is associated with the special product leading to the difference of two squares.

$$(3\sqrt{2}-2\sqrt{3})(3\sqrt{2}+2\sqrt{3})=(3\sqrt{2})^2-(2\sqrt{3})^2=18-12=6$$

EXERCISES 9-5

Perform the indicated operations and simplify results.

1. $5\sqrt{x}\cdot 3\sqrt{x}$
2. $4\sqrt[3]{9}\cdot\frac{1}{12}\sqrt[3]{3}$
3. $\sqrt{abc}\cdot\sqrt{a^2b}$
4. $\sqrt{6x}\cdot\sqrt{3x}-\sqrt{2x^2}$
5. $\dfrac{-\sqrt{35}\cdot\sqrt{35}}{7}$
6. $(\sqrt{2x}-1)(\sqrt{2x}+1)$
7. $(\sqrt{11}-\sqrt{5})(\sqrt{11}+\sqrt{5})$
8. $(\sqrt{11}-\sqrt{5})^2$
9. $(\sqrt{11}+\sqrt{5})^2$
10. $(3\sqrt{x}-3)(3\sqrt{x}+3)$
11. $\sqrt[3]{16}\cdot\sqrt[3]{16}$
12. $\sqrt[3]{9x^2}\cdot\sqrt[3]{3x}$
13. $(5)^{1/2}(15)^{1/2}$
14. $(3x)^{1/3}(9x^2)^{1/3}$
15. $\sqrt[3]{25}\cdot\sqrt{5}$ (use fractional exponents)
16. $\sqrt[3]{25}\div 5$
17. $\dfrac{3\sqrt{6}}{6\sqrt{3}}$
18. $\dfrac{a^2\sqrt[3]{7}}{\sqrt[3]{49a^6}}$
19. $\dfrac{a^3b}{ab^3}$
20. $\dfrac{\sqrt{26}}{\sqrt{13}}$
21. $\dfrac{6}{\sqrt{6}-3}$
22. $\dfrac{-6}{\sqrt{6}+3}$
23. $\dfrac{\sqrt{6}}{\sqrt{6}-a}$
24. $\dfrac{11}{2\sqrt{11}-5}$
25. $\dfrac{2\sqrt{3}+3\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$
26. $\dfrac{3\sqrt[3]{3}-2\sqrt[3]{81}}{3\sqrt[3]{81}+2\sqrt[3]{3}}$
27. $\dfrac{a-b}{\sqrt{a}-\sqrt{b}}$
28. $\dfrac{\sqrt{ab}-\sqrt{ab}}{\sqrt{ab}}$
29. $\dfrac{a^2-b^2}{\sqrt{a}+\sqrt{b}}$
30. $\dfrac{\sqrt{x+y}}{\sqrt{x}+\sqrt{y}}$
31. $\sqrt[3]{x-y}\cdot\sqrt[3]{x^2-2xy+y^2}$

32 $\dfrac{\sqrt{4a^2 + 8ab + 4y^2}}{2\sqrt{a + y}}$

33 $\dfrac{\sqrt[3]{27 + 27b + 9y^2 + y^3}}{\sqrt{y^2 + 6y + 9}}$

34. $\dfrac{\sqrt{2a} + b - \sqrt{2a}}{\sqrt{2a} - b + \sqrt{2a}}$

35. $\dfrac{\sqrt{a - b}}{\sqrt{(a - b)^3}} - \dfrac{\sqrt{a + b}}{\sqrt{(a + b)^3}}$

9-7 IMAGINARY NUMBERS

If the number system were developed presently, in all probability the term imaginary number would not be included An imaginary number is as much a reality as any other number that the technician may come to know

In the analysis of $a - c$ circuits, this concept is very much "alive" and is handled with the same fundamental mathematical operations as are applied to most expressions

An imaginary number is defined as the square root of a negative number There exists no real number, such that when multiplied by itself it will lead to a negative product Thus the following quantities cannot be simplified or evaluated without introducing another mathematical concept

$$\sqrt{-4},\ \sqrt[4]{-16},\ \text{etc}$$

This concept is represented as follows

$$i = j = \sqrt{-1} \quad \text{(an imaginary unit)}$$

where the letter i is used by mathematicians and j by technicians and engineers (i is used to designate current in electronics)

Thus the square root of a negative number can be defined as the product of the (real) principal root and j or i

Hence,

$$\sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = \sqrt{-1} \cdot 2 = j2 = i2$$

where $j2$ or $i2$ is the imaginary root of $\sqrt{-4}$ ($i2$ is the same as $2i$)

The symbol i (or j) is treated as any other literal symbol during an algebraic operation

$$(2x)(3x) = 6x^2, \qquad (2i)(3i) = 6i^2$$
$$2j + 3j = 5j, \qquad i2 - 6i + 2a = -4i + 2a, \text{ or } -i4 + 2a$$

Some functional relationships concerning i or j will be developed

$$i^0 = j^0 = 1$$
$$i = j = \sqrt{-1}$$
$$i^2 = j^2 = (\sqrt{-1})^2 = -1$$
$$i^3 = j^3 = i^2 i = (-1)\sqrt{-1} = -\sqrt{-1} = -i$$
$$i^4 = j^4 = j^2 j^2 = (-1)(-1) = 1$$

In general form an imaginary number is defined as:

$$\sqrt{-a} = i\sqrt{a} \text{ or } j\sqrt{a}$$

EXAMPLE 9-O:

Find the difference: $\sqrt{-64} - \sqrt{-25}$.

Solution:

Imaginary numbers are combined algebraically (or arithmetically) only when they are of the form $i\sqrt{a}$ rather than $\sqrt{-a}$.

Thus,

$$\sqrt{-64} = \sqrt{(-1)(64)} = i8 \text{ or } 8i$$
$$\sqrt{-25} = \sqrt{(-1)(25)} = i5 \text{ or } 5i$$

Therefore,

$$\sqrt{-64} - \sqrt{-25} = 8i - 5i = 3i \text{ or } i3$$

EXAMPLE 9-P:

Multiply $(\sqrt{-5})(\sqrt{-9})$.

Solution:

$$\sqrt{-5} = j\sqrt{5}; \qquad \sqrt{-9} = j3$$

Thus,

$$(\sqrt{-5})(\sqrt{-9}) = (j\sqrt{5})(j3) = 3j^2\sqrt{5} = 3(-1)\sqrt{5} = -3\sqrt{5}$$

The distinction among the following expressions must be recognized.

$$(\sqrt{-5})(\sqrt{-9}) \neq \sqrt{(-5)(-9)}$$
$$(\sqrt{-5})(\sqrt{-9}) = -3\sqrt{5}$$
$$\sqrt{(-5)(-9)} = \sqrt{45} = 3\sqrt{5}$$

The laws of radicals were developed to deal with real numbers; $\sqrt{-5}$ and $\sqrt{-9}$ are not real numbers, whereas (-5) and (-9) in the expression $\sqrt{(-5)(-9)}$ are factors representing a real number. Thus, $(\sqrt{-a})(\sqrt{-b}) \neq \sqrt{(-a)(-b)}$.

EXAMPLE 9-Q:

Divide

$$\frac{\sqrt{-45}}{\sqrt{-9}}.$$

Solution:

$$\frac{\sqrt{-45}}{\sqrt{-9}} = \frac{\sqrt{(-1)(45)}}{\sqrt{(-1)(9)}} = \frac{i\sqrt{45}}{i\sqrt{9}} = \frac{i3\sqrt{5}}{i3} = \sqrt{5}$$

where $\frac{i3}{i3} = 1$.

Perform the indicated operations and simplify, wherever possible

1. $(j^2)(j)$
2. $\sqrt{-36} + \sqrt{-27} - \sqrt{8}$
3. $\sqrt{-50} + \sqrt{-72} - \sqrt{2}$
4. $\sqrt{-288} + \sqrt{-338} - \sqrt{-378}$
5. $(\sqrt{-7})(\sqrt{-7}) + 49$
6. $(6j)(3j)$
7. $(-3\sqrt{-4})(-2\sqrt{-3})$
8. $\sqrt{-432} - \sqrt{-108}$
9. $(\sqrt{-25} - \sqrt{-6})(\sqrt{-25} + \sqrt{-6})$
10. $\sqrt{-625} - 5j$
11. $(12j \cdot 9j) - 27j^2$
12. $\sqrt{-64} - \sqrt{-8}$
13. $\sqrt{-32} \cdot \sqrt{-32} + \sqrt{32}$
14. $(j7)(j5)(j2)$
15. $(i15 - i3)\sqrt{-5}$
16. $\sqrt{-x^2} \cdot \sqrt{-4x^2}$
17. $\sqrt{\dfrac{-64}{25}}$
18. $\sqrt{-i^2}$
19. $(-\sqrt{-35})(-\sqrt{-35})$
20. $(j5 - 1)(j5 + 1)$

9 8 COMPUTING SQUARE ROOT

The square root of a number may be determined by using the calculator, slide rule, logarithms, tables, or the arithmetic process The last procedure will be demonstrated with the use of several examples

EXAMPLE 9-R

Determine $\sqrt{1{,}310\,44}$

Solution

Initially, the digits of the given number are grouped in pairs on both sides of the decimal point

$$13'\ \ 10\ \ '44$$

The arithmetic operation is developed around pairs of digits If there are an odd number of digits to the right, a zero is added in keeping with the principle Zeros are also added to meet accuracy requirements If there are an odd number of digits to the left, the single digit remains and this is the only time that a single digit is warranted within the context of grouping

The first step is somewhat unique with respect to subsequent steps, the object here is to find the largest number whose square is less than the digits (digit) under consideration In terms of the illustrative example, it is apparent that the largest perfect square, less than 13, is 9, where $(3)^2 = 9$ Procedurally, this is represented accordingly

$$\begin{array}{rl} & \ \ 3 \\ & \sqrt{13\ 10\ 44} \\ \text{Subtract} & \ \ \underline{\ 9}\ \downarrow \\ & \ \ \ 4\ 10 \end{array}$$

The number 3 is placed above the first grouped set, outside the radicand, and its square is placed below the same set, as indicated. Next, subtract 9 from 13 and bring down the immediate pair of digits to go along with the difference.

After the first step, the procedure changes and becomes somewhat like long division.

A divisor for the remainder 410 is determined in the following step. This quotient then becomes the second digit of the root. To find the quotient, first double the incomplete root and multiply by 10. This product becomes a **trial divisor.**

$$(2)(3)(10) = 60$$

$$\begin{array}{r} 6 \\ 60\overline{)410} \\ \underline{360} \end{array}$$

The **actual** or **complete divisor** is equal to the trial divisor plus the partial quotient.

$$60 + 6 = 66$$

The complete divisor is now divided into the remainder, 410, whereby the quotient becomes the second digit of the partial root.

$$\begin{array}{rr} & 3\ \ 6 \\ & \sqrt{13\ 10\,.44} \\ & 9 \\ 66 & \overline{)4\ 10} \\ & \underline{396} \\ \text{Subtract} & 1{,}444 \end{array}$$

The procedure of the previous step is repeated, using this time the difference, 1,444.

$$\text{Trial divisor} = 2(\text{partial root})(10) = 2(36)(10) = 720$$

$$\begin{array}{r} 2 \\ 720\overline{)1{,}444} \\ \underline{1{,}440} \end{array}$$

$$\text{Actual divisor} = \text{trial divisor} + \text{quotient} = 720 + 2 = 722.$$

Thus,

$$\begin{array}{rl} & 3\ \ 6\,.2 \\ & \sqrt{13\ 10\,.44} \\ & 9 \\ 66 & \overline{)4\ 10} \\ & 3\,96 \\ 722 & \overline{)\ 14\ 44} \\ & \underline{14\ 44} \\ & 0 \text{ Remainder} \end{array}$$

which completes the computation: $\sqrt{1{,}310.44} = 36.2$.

Check:

If 36 2 is the principal square root of 1310 44, it follows that (36 2)(36 2) = 1310 44

$$
\begin{array}{r}
36\ 2 \\
\underline{36\ 2} \\
7\,24 \\
217\,2 \\
\underline{1\,086} \\
1{,}310\ 44
\end{array}
$$

EXAMPLE 9-5

Find the square root of 695 41 carried to two decimal places

Solution

Start by pairing off the digits and adding pairs of zeros to assure two decimal places If accuracy is desired to two decimal places, the computation should be carried out to involve the third decimal position, in conformity with the procedure of rounding off

695 41 paired '6 '95 '41 00 '00

The largest number whose square is less than 6 is of course (2)(2) = 4 Thus,

$$
\begin{array}{rl}
 & \phantom{\sqrt{}}2 \\
 & \sqrt{6\,95\ 41\,00\,00} \\
\text{Subtract} & \underline{4}\downarrow \\
 & 2\,95
\end{array}
$$

Double the first digit of the partial root, and multiply by 10 to form a trial divisor for the initial remainder (295)

$$
2(2) \times 10 = 40, \qquad 40\overline{)295}^{\,7} \quad \underline{280}
$$

Thus, it appears that the complete divisor is

$$40 + 7 = 47$$

But,

$$
47\overline{)295}^{\,7} \quad \underline{329}
$$

which means that the actual divisor is $40 + 6 = 46$, rather than 7, since $47 \times 7 = 329$, where $329 > 295$

Using 46 as the divisor and proceeding

$$
\begin{array}{rl}
 & \phantom{\sqrt{}}2\ \ 6 \\
 & \sqrt{6\,95\ 41\,00\,00} \\
 & \ \ 4 \\
 & 46\overline{)2\,95}\downarrow \\
\text{Subtract} & \underline{2\,76} \\
 & \ \ 19\,41
\end{array}
\qquad \text{(bring down the next pair of digits)}
$$

Again double the incomplete root, 26, and multiply by 10 to find the second trial divisor: $2(26) \times 10 = 520$

$$\begin{array}{r} 3 \\ 520\,\overline{)\,1{,}941} \\ 1{,}560 \\ \hline 381 \end{array}$$

Complete divisor $= 520 + 3 = 523$

$$\begin{array}{r} 3 \\ 523\,\overline{)\,1{,}941} \\ 1{,}569 \\ \hline 372 \end{array}$$

Continuing:

$$\begin{array}{rl} & \;2\;\;6\,.\;3 \\ & \sqrt{6\;95\,.41\;00\;00} \\ & \;4 \\ 46 & \overline{)\,2\;95} \\ & \;2\;76 \\ 523 & \overline{)\;\;19\;41} \\ & \;\;\;15\;69 \\ \text{Subtract} & \;\;\;\;\;3\;72\;00 \end{array}$$

The third trial divisor is equal to $2(263) \times 10 = 5{,}260$

$$\begin{array}{r} 7 \\ 5{,}260\,\overline{)\,37{,}200} \\ 36{,}820 \\ \hline \end{array}$$

Complete divisor $= 5{,}260 + 7 = 5{,}267$

$$\begin{array}{r} 7 \\ 5{,}267\,\overline{)\,37{,}200} \\ 36{,}869 \\ \hline \end{array}$$

Incorporating this result within the next step leads further to:

$$\begin{array}{rl} & \;2\;\;6\,.\;3\;\;7 \\ & \sqrt{6\;95\,.41\;00\;00} \\ & \;4 \\ 46 & \overline{)\,2\;95} \\ & \;2\;76 \\ 523 & \overline{)\;\;19\;41} \\ & \;\;\;15\;69 \\ 5{,}267 & \overline{)\;\;\;\;3\;72\;00} \\ & \;\;\;\;\;3\;68\;69 \\ \hline & \;\;\;\;\;\;\;3\;31\;00 \end{array}$$

Final trial divisor:

$$2(2{,}637) \times 10 = 52{,}740$$

$$\begin{array}{r} 0.+ \\ 52{,}740\,\overline{)\,33{,}100.} \end{array}$$

which would give a quotient less than 1 Therefore, the third decimal place would be reflected by the number 0

Thus,

$$\sqrt{695\ 41} = 26\ 37$$

Check:

$$(26\ 37)(26\ 37) = 695\ 38$$

which indicates that the resulting computation is only an approximation

EXERCISES 9-7

Find, by arithmetic computation, the square roots of the following numbers Carry out to two decimal places (Ex 1-10)

1. 169 00   *2.* 6 25

3. 1,288.81   *4.* 3,721

5. 29,000 00   *6.* 173 90

7. 0 2601   *8.* 0 576

9. 13 31   *10.* 1 48

REVIEW EXERCISES 9-8

Simplify the given expression (Ex 1-10)

1. $\sqrt{-169}$ *2.* $(-686)^{1/3}$

3. $\sqrt{72}\cdot\sqrt{-72}$ *4.* $\sqrt[3]{\frac{9}{16}}$

5. $\left(\frac{9}{16}\right)^{1/2}$ *6.* $-\left(-\frac{4}{9}\right)^{1/3}$

7. $(25)^{3/2}$ *8.* $\left(-\frac{1}{8}\right)^{2/3}$

9. $\left(\frac{2}{3}\right)^{-2}\left(\frac{2}{3}\right)^{3}$ *10.* $\frac{-\sqrt[3]{-27x^3}}{\sqrt[3]{27x^5}}$

Perform the indicated operations and simplify accordingly (Ex 11-28)

11. $3\sqrt{5} - 5\sqrt{5} + 2\sqrt{5}$ *12.* $7\sqrt{12} - 4\sqrt{18} + a\sqrt{72}$

13. $4\sqrt[3]{16a^4} - 3a\sqrt[3]{81a} + 5\sqrt{108a^3}$

14. $3\sqrt{7}(4a\sqrt{147} - 7\sqrt{216})$ *15.* $(x^2 - y^2)^{1/2}(x^2 + y^2)^{1/2}$

16. $3\sqrt{18a^3b} - 6\sqrt{2ab^3}$ *17.* $\frac{9a}{\sqrt{a}-2}\cdot\frac{3a}{\sqrt{a}+2}$

18. $\frac{3x-y}{\sqrt{x}-\sqrt{y}}\cdot\frac{3x+y}{\sqrt{x}+\sqrt{y}}$ *19.* $\sqrt{15} - (\sqrt{3} - 3)$

20. $(a^{2/3} - b^{2/3})(a^{2/3} + b^{2/3})$ *21.* $(\sqrt{T} + 5) - (\sqrt{T} - 5)$

22. $(\sqrt{-7} + 5)(\sqrt{-7} - 5)$

23. $\sqrt{75a^3b^5} \div \sqrt{5ab^3}$

24. $2a\sqrt{-338} + 4a\sqrt{-378} - 3a\sqrt{-672}$

25. $\dfrac{\sqrt{10}}{\sqrt{x} - 4} + \dfrac{\sqrt{13}}{\sqrt{x} + 4}$

26. $\sqrt[3]{(4x^2 - 12x + 9)^2}$

27. $\sqrt[3]{(4x^2 - 12x + 9)^2} \div \sqrt[3]{8x^3 - 36x^2 + 54x^2 - 27}$

28. $\dfrac{x^3 - y^3}{\sqrt{x} - \sqrt{y}}$

Compute the square root and carry out to three decimal places.

29. $\sqrt{834.76}$

30. $\sqrt{3,649.81}$

31. TV signals travel in a straight line, hence do not follow the curvature of the earth. The range, R, in miles, of a TV signal can be computed by the formula $R = 1.23\sqrt{h}$, where h, the height of the transmitter, is given in feet (Fig. 9-1). Find R if $h = 1{,}600$ ft.

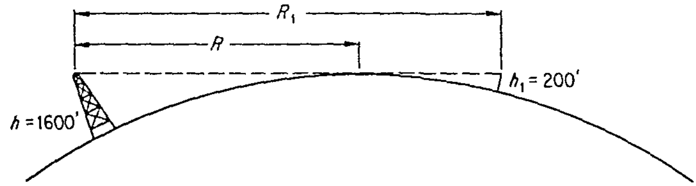


Figure 9-1

32. Find R if the height of the receiving antenna is 200 ft above the horizon (Fig. 9-1).

33. The period of a pendulum, the time it takes it to complete one complete cycle, is given by the formula:

$$t = 2\pi\sqrt{\frac{l}{g}},$$

where $g = 32.2$ ft/sec^2, t is in seconds, and l, the length of the pendulum, is in feet (Fig. 9-2).

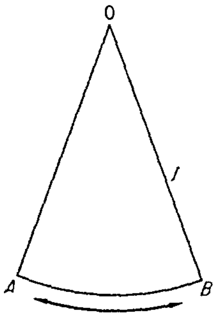


Figure 9-2

(a) Find t if $l = 36.0$ in.
(b) Find t if $l = 32.2$ ft.

34 The mass of a body increases with velocity. This principle is stated by the formula

$$m = \frac{m_i}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m is the mass of the body (grams) when the velocity is v (miles/sec). The constant c is the velocity of light: $c = 3 \times 10^8$ miles/sec and m_i is the mass when the body is at rest.

a. Find m if $v = 10$ miles/sec and $m_i = 100$ g.

b. Find m if $v = 1{,}000$ cm/sec and $m_i = 1$ kg.

CHAPTER 10

Linear Equations

An *equation* is a statement of equality involving two expressions. In mathematics this relates to literal numbers and various other quantities, usually numbers. In technology, an equation relating to certain engineering-scientific concepts is called a *formula*.

$$x - 3 = 7, \quad x^2 + 2x + 1 = 0 \qquad \text{(equations)}$$

$$S = \frac{Mc}{I}, \quad Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad \text{(formulas)}$$

In the equation $x - 3 = 7$, x is called the *variable*, since it takes on different meaning as conditions vary. For the two equations listed, $x = 10$ in the first and $x = -1$ in the second. Thus, x changes in numerical value to meet stated conditions.

10-1 EQUATIONS AND IDENTITIES

Equations are classified as *conditional* or *identical*. If an equation holds true for only a limited number of values, it is called a *conditional equation*. A conditional equation is referred to as an equation.

An example of a conditional equation is $x + 2 = 5$. Upon examination, we see that this condition is true only when $x = 3$, where $3 + 2 = 5$, and the equation is satisfied.

The procedure for finding the value of the variable, or unknown, is called *solving the equation*.

An equation that holds true for all values of the variable(s) for which it is defined is referred to as an *identity*.

$$x^2 - y^2 = (x - y)(x + y)$$

This condition is satisfied for all values of x and y. Several values will be tried, as a matter of demonstration.

$$x^2 - y^2 = (x - y)(x + y)$$

Let $x = 5$ and $y = 4$
Then $(5)^2 - (4)^2 = (5 - 4)(5 + 4)$, where

$$25 - 16 = (1)(9) \text{ and } 9 = 9$$

Let $x = -3$ and $y = 2$
Then $(-3)^2 - (2)^2 = (-3 - 2)(-3 + 2)$ where

$$9 - 4 = (-5)(-1) \text{ and } 5 = 5$$

Finally, let $x = 0$ and $y = -7$
Then $(0)^2 - (-7)^2 = (0 + 7)(0 - 7)$ where

$$0 - 49 = (7)(-7) \text{ and } -49 = -49$$

The principle of equivalent fractions provides another example of identities

$$\frac{1}{\frac{1}{x} - \frac{1}{y}} = \frac{xy}{y - x} \qquad (x \neq 0 \text{ and } y \neq 0)$$

Identities are used most frequently in trigonometry The symbol for identity is $\equiv$, to distinguish it from equation $=$

Every identity can be classified as an equation, however, not every equation meets the criterion of an identity

10-2 SOLVING EQUATIONS

Several axioms involving the properties of equations were discussed in Sec 5-6, and it might be well to glance over that portion of the text that complements the current material

An equation states that everything on the left of the symbol, ($=$), is equal to everything on the right or both sides are equal to each other

The procedure for solving an equation usually involves the process of changing the original equation to an equivalent equation This can be accomplished in one or more of the following ways

1 Adding the same numerical quantity to both sides of the equation,
2 Subtracting the same numerical quantity from both sides of the equation,
3 Multiplying both sides by the same factor, and
4 Dividing both sides by the same quantity, where the quantity is other than zero

Given $(5 + 2) = (4 + 3)$, it then follows,
if 3 is added to both sides, $(5 + 2) + 3 = (4 + 3) + 3$ and $10 = 10$

if 3 is subtracted from both sides, $(5 + 2) - 3 = (4 + 3) - 3$ and $4 = 4$

if both sides are multiplied by 7, $7(5 + 2) = 7(4 + 3)$ and $49 = 49$

if both sides are divided by -14, $(5 + 2)/-14 = (4 + 3)/-14$ and $1/-2 = 1/-2$

In all cases the equation is balanced.

None of the operations mentioned above will change the value of the original equation, provided the quantity used in multiplication or division does not contain the *unknown* or *variable* that is usually denoted by x, y, or z. Multiplication of both sides of an equation by a factor containing the variable, with few exceptions, may introduce additional roots referred to as *extraneous roots*. Division with factors containing the variable usually takes away roots, termed *vanishing roots*.

An equation is considered solved when the variable is defined and the solution satisfies the original equation.

$$x = 5, \quad x = a + b, \quad y = 7, \quad z = -8$$

In practice the variable is on the left whereas the numerical or other quantity is on the right; however, $5 = x$ and $x = 5$, are identical statements, indicating that the position of the variable is only a matter of preference.

The technique for solving an equation involves collecting all the variables on one side of the equation and all the numerical or other similar quantities on the opposite side. In other words, similar terms are on respective sides. These terms are then combined algebraically and the equation is solved.

10-3 LINEAR EQUATIONS

An equation of the form: $ax + c = 0$ is a *linear equation in one variable* (x). An equation of the form $ax + by + c = 0$ is a *linear equation in two variables* (x and y).

When the **highest exponent** of the variable in an equation is **1**, the equation is called a **linear equation**: $2x + 3y - 5z + 10 = 23$. When the exponent is 2, the equation is classified as a *quadratic*: $3x^2 + 2xy - 7 = 0$.

The equation $3xy = 4$ is also called a *quadratic* because the sum of the exponents of x and y is 2. Furthermore, $x^3 - 3x^2y + 3xy^2 - y = 0$ is called a *cubic*, whereas, $5x^4 - 3x^3y + 2y^2 + 8 = 0$ is considered a *quartic*.

The equations to be studied in this section will be linear equations in one variable: $2x + 3 = 9$, $5y - 13 = 12$, $12z = 5 - 6z, \ldots$.

EXAMPLE 10-A:

Solve the following equations: (a) $3x + 2x - 4 = 6 + 5$; (b) $7x/2 + 3/5 = 3x - 4$; (c) $6(y - 2) + 10(y + 2) = 15(y - 1)$.

(a) $3x + 2x - 4 = 6 + 5$

Solution

$3x + 2x - 4 = 6 + 5$ Collect like terms first

$5x - 4 = 11$ Next, add $+4$ to both sides

$\underline{\quad + 4 = +4}$

$5x \quad = 15$ Divide both sides by 5

$\frac{5x}{5} = \frac{15}{5}$, where $x = 3$

Thus, $x = 3$ is the solution of the equation $3x + 2x - 4 = 6 + 5$

If $x = 3$ is the solution, it must satisfy the *original equation* This can be verified by substituting 3 for x in the equation $3x + 2x - 4 = 6 + 5$ (Whenever x appears, replace it with its numerical value 3)

$$3x + 2x - 4 = 6 + 5$$

$$3(3) + 2(3) - 4 = 6 + 5, \text{ or } 9 + 6 - 4 = 11$$

Furthermore, $15 - 4 = 11$, and it is apparent that the equation is satisfied

(b) $\frac{7x}{2} + \frac{3}{5} = 3x - 4$

Solution

$\frac{7x}{2} + \frac{3}{5} = 3x - 4$ Clear fractions by multiplying through by LCD, which is equal to 10

$10\left(\frac{7x}{2}\right) + 10\left(\frac{3}{5}\right) = 10(3x) - 10(4)$

or,

$35x + 6 = 30x - 40$ Next, subtract $30x$ and 6 from both sides of the equation

$35x + 6 = 30x - 40$

$\underline{30x + 6 = 30x + \ 6}$

$5x \quad = \quad -46$ Divide both sides by 5

$\frac{5x}{5} = \frac{-46}{5}$, where $x = -\frac{46}{5}$ (solution or root)

(c) $6(y - 2) + 10(y + 2) = 15(y - 1)$

Solution

Remove parentheses and collect like terms

$$6(y - 2) + 10(y + 2) = 15(y - 1)$$

$$6y - 12 + 10y + 20 = 15y - 15$$

or,

$$16y + 8 = 15y - 15$$

Next, add $-15y$ and -8 to both sides (or subtract $15y$ and 8).

$$\begin{array}{rcr} 16y + 8 &=& 15y - \quad 15 \\ -15y - 8 &=& -15y - \quad 8 \\ \hline y &=& -23 \end{array}$$

which is the solution, or root, of the given equation.

Checking:

$$6(y - 2) + 10(y + 2) = 15(y - 1)$$
$$6(-23 - 2) + 10(-23 + 2) = 15(-23 - 1).$$

Furthermore,

$$6(-25) + 10(-21) = 15(-24),$$

or

$$-150 - 210 = -360$$

which indicates that the equation is satisfied.

It should be pointed out that in *checking* the *solution* (*root*), only the *original equation* should be used, *not* an *equivalent equation.* An equivalent equation may contain only some of the roots.

Perhaps it might be advisable to investigate those conditions under which roots may be lost or added. This is a deviation from the topic of linear equations but rather appropriate. The subject of *extraneous roots* and *vanishing roots* will become more meaningful as the technician gains knowledge of quadratics.

Starting with the equation $x - 2 = 2$, it is apparent that the root or solution is $x = 4$, where $4 - 2 = 2$ (check).

Now, multiplying both sides of the equation by $(x - 2)$ leads to $(x - 2)(x - 2) = 2(x - 2)$, which simplifies further to $x^2 - 4x + 4 = 2x - 4$. Collecting terms leads to the quadratic $x^2 - 6x + 8 = 0$, whose roots are $x = 4$ and $x = 2$.

These can be verified by substitution:

$$x = 4, \quad (4)^2 - 6(4) + 8 = 0 \quad \text{and} \quad 16 - 24 + 8 = 0$$
$$x = 2, \quad (2)^2 - 6(2) + 8 = 0 \quad \text{and} \quad 4 - 12 + 8 = 0$$

Only those expressions, however, that *satisfy* the *original equation* can be *considered roots*. Checking back with the original equation leads to the following observation:

When $x = 4$, $x - 2 = 2$ becomes $4 - 2 = 2$ and the equation is satisfied.

When $x = 2$, $x - 2 = 2$ becomes $2 - 2 \neq 2$ and the equation is not satisfied Thus, $x = 2$ is not a real root, but rather is an extraneous root

The equation $x^2 - 3x + 2 = 0$, whose real roots are $x = 2$ and $x = 1$, will be used to illustrate a situation leading to a vanishing root

Dividing the given equation by the factor $(x - 2)$ leads to

$$\frac{x^2 - 3x + 2}{x - 2} = \frac{0}{x - 2}$$

which simplifies finally to, $x - 1 = 0$, where $x = 1$

Thus, the operation just demonstrated leads to a solution containing only one root It can be verified that the given equation has two roots Hence, dividing an equation by an expression containing the variable or unknown may lead to a loss of a root, termed the vanishing root

The topic of extraneous and vanishing roots is not completed in this section The exposure to this principle was intended simply to make the student aware of it

EXAMPLE 10 B

Solve the equation $\dfrac{1}{x + 3} + \dfrac{2x}{x - 3} = 2$

Solution

In order to remove fractions, it is necessary to multiply through by the LCD, which turns out to be $(x + 3)(x - 3) = x^2 - 9$ Multiplying by the LCD (usually) will not introduce additional roots

$$(x + 3)(x - 3)\left(\frac{1}{x + 3}\right) + (x + 3)(x - 3)\left(\frac{2x}{x - 3}\right) = 2(x^2 - 9)$$

which becomes

$$x - 3 + 2x(x + 3) = 2x^2 - 18$$
$$x - 3 + 2x^2 + 6x = 2x^2 - 18$$
$$2x^2 + 7x - 3 = 2x^2 - 18$$

Subtracting $2x^2$ from both sides leads to the expression $7x - 3 = -18$ Adding 3 to both sides reduces the equation to the form $7x = -15$, from which $x = -15/7$

Thus, the solution of the given equation is $x = -15/7$

The root may be verified or checked by substituting back into the original equation

$$\frac{1}{x + 3} + \frac{2x}{x - 3} = 2$$

Substituting for x and simplifying

$$\frac{1}{\frac{-15}{7}+3}+\frac{2\left(\frac{-15}{7}\right)}{\frac{-15}{7}-3}=2$$

$$\frac{1}{\frac{-15+21}{7}}+\frac{\frac{-30}{7}}{\frac{-15-21}{7}}=2$$

$$\frac{7}{6}+\frac{\frac{-30}{7}}{\frac{-36}{7}}=2$$

$$\frac{7}{6}+\frac{-30}{-36}=2$$

and

$$\frac{7}{6}+\frac{5}{6}=\frac{12}{6}=2$$

The equation is satisfied; hence, $x = -15/7$ is a root.

The procedure, outlined below, may serve as a guide to solving equations.

1. Remove parentheses.
2. Clear all fractions by multiplying through by the LCD.
3. Collect like terms.
4. Transpose, such that the variable is on one side and the numbers or other quantities are on the other side.
5. Divide both members of the equation by the coefficient of the variable.
6. Check root(s) by substituting back in the original equation.

EXERCISES 10-1

Classify as equations or identities (Ex. 1-8).

1. $(x-2)+(3x-5)=3x^2-11x+10$

2. $3x+5=2x+4+1$

3. $x^2+2x+1=(x+1)(x+1)$

4. $3x^2-3x+15=2x^2-3x+15-4x^2$

5. $\dfrac{x}{x-1}-\dfrac{x}{x+1}=\dfrac{2x}{x^2-1}$

6. $16x-(13+5x)=13+5x-16x$

7. $\dfrac{3x^2-x-6}{x-2}=\dfrac{x-2}{3x+5}$

8. $(x - 4) - (x - 4) = 2(x - 4)$

Determine if the given values are roots of the respective equations (Ex 9-16)

9. $(x + 3) - (2x + 7) = 3,\ x = -7$

10. $3(x + 4) = 2(x - 6),\ x = 6$

11. $3(2x + 5) - 5x = -4(x - 3) - 7,\ x = -2$

12. $x^2 - 5x + 7 = 7,\ x = 0,\ x = 5$

13. $\dfrac{2x}{x - 3} + \dfrac{x}{x + 3} = 3,\ x = 3,\ x = -3$

14. $5x^2 + 2x - 3 = (5x - 3)(x + 1),\ x = 0,\ x = \dfrac{3}{5}$

15. $\sqrt{2x - 1} + \sqrt{6x - 5} = \sqrt{9x + 19},\ x = +5,\ x = -5$

16. $(2x - 3)(4x^2 - 12x + 9) = 8x^3 - 36x^2 + 54x - 27,$
$x = 0,\ x = -1,\ x = -3$

Solve the following equations and check roots (solution)

17. $x + 1 = 7$

18. $x - 1 = 7$

19. $2x + 1 = 11$

20. $2x + 1 = -11$

21. $3x = 16 + 2$

22. $4x + 2 = 17$

23. $4x + 3 - 2x + 6 = x - 7$

24. $\dfrac{3x}{4} = \dfrac{3}{4}$

25. $\dfrac{-21}{x} = -3$

26. $3x - 11 - 2x = 2(x - 6) + 1$

27. $x - \dfrac{3}{4} = \dfrac{3x}{8} + 2$

28. $3y + 5a = y - 7a$

29. $6c = 2\pi d$

30. $7ax - 15a + 2ax = 3a(x + 2) + 9a$

31. $3y = 21a - 6b$

32. $4ay - 8by = a^2 - 4ab + 4b^2$

33. $4y(2a - b) = 4a^2 - b^2$

34. $9ax = 9a^2 + 15x - 25$

35. $10ax - 8bx + 15ax - 12bx = 5(5a - 4b)$

36. $(x - 3)(x + 5) = (x - 5)(x + 2)$

37. $(x^3 - 8) + 2x(x - 3) = x^2(x + 2) + 3x + 10$

38. $\dfrac{2z}{z - 3} + 4 = 0$

39. $\dfrac{3}{w+1}+\dfrac{3}{w-2}=0$

40. $\dfrac{1}{r-3}-\dfrac{2}{r-2}=\dfrac{3}{r-3}$

41. $\dfrac{3x+1}{x-4}-\dfrac{x-2}{x+4}=\dfrac{x(2x+17)}{x^2-16}$

42. $\dfrac{2}{y}-\dfrac{3}{2y-3}=\dfrac{1}{2y+3}$

10-4 FORMULAS

Formulas are equations that express, in symbols, various mathematical and scientific-engineering principles. Formulas are developed through a combination of procedures involving definitions, experiments, or dimensional analysis. The solution of a formula or an element contained in the formula involves the same procedures established for the solution of equations.

EXAMPLE 10-C:

$$\text{Given } I=\frac{nE}{Re+nRi}, \text{ solve for } n.$$

Solution:

Multiply both sides of the formula by $Re+nRi$ to clear fractions:

$$(Re+nRi)I=\frac{nE}{(Re+nRi)}(Re+nRi)$$

simplifies to

$$IRe+nIRi=nE$$

Collect terms containing n

$$nE-nIRi=IRe$$

Solve for n

$$n(E-IRi)=IRe$$

$$n=\frac{IRe}{E-IRi}$$

Thus the original formula is now expressed in terms of n.

In terms of a practical solution, the term n can be given further meaning. Find n if $I=5$ amps, $Re=5$ ohms, $Ri=0.25$ ohm, and $E=7.5$ v.

Substituting in the formula, accordingly, leads to the solution

$$n=\frac{IRe}{E-IRi}=\frac{5(5)}{7.5-5(0.25)}=\frac{25}{7.5-1.25}=\frac{25}{6.25}=4 \text{ cells}$$

Solve for that element of the formula, as required, and find its value

1. The linear expansion owing to temperature change is $e = \alpha l(T_2 - T_1)$ Solve for l, the original length

2. Find l if $T_2 = 92°F$, $T_1 = 62°F$, $\alpha = 6\ 0 \times 10^{-6}$, and $e = 1\ 8$ in (l is measured in inches)

3. Capacitive reactance is $X_c = 1/2\pi fC$ Solve for the frequency, f (cycles per second)

4. Find f if $X_c = 200$ ohms and $C = 12\ 5 \times 10^{-6}$ f

5. The equation for the force acting on the base of a container is $F = AhD$ Solve for h

6. Determine the maximum height, h, that a liquid with a density of $D = 40$ lb/ft³ can reach without exerting a force, F, greater than 400,000 lbs on the base of a container whose diameter is 20 ft (A = area)

7. The joint resistance of three parallel resistors is expressed by the following formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

From the given formula, supply all the algebraic steps that lead to the equivalent formula

$$R = \frac{R_1R_2R_3}{R_1R_2 + R_1R_3 + R_2R_3}$$

8. Find the joint resistance of three resistors in parallel if $R_1 = 9$ ohms, $R_2 = 12$ ohms, and $R_3 = 15$ ohms

The following expressions are additional formulas that appear in the area of technology Solve for the indicated term or element

Formula	Solve for
9. $°C = \frac{5}{9}(°F - 32°)$	°F
$°C = -40°$	
10. $v = v_i + gt$	t
$v = 278$ ft/sec, $v_i = 20$ ft/sec, $g = 32\ 2$ ft/sec²	
11. $A = \frac{h}{2}(a + b)$	a
$A = 60$ in ², $h = 6$ in , $b = 8$ in	
12. $\frac{PV}{T} = \frac{pv}{t}$	v

$P = 400$ mm, $V = 500\ l$, $p = 500$ mm

$T = 300°K$, $t = 360°K$

13. $\dfrac{1}{D_o} + \dfrac{1}{D_i} = \dfrac{1}{f}$ D_i

$D_o = 144$ in., $f = 16$ in.

14. $W = \dfrac{2PR}{R - r}$ r

$P = 200$ lb, $R = 18$ in., $W = 600$ lb

15. $R_1 = \dfrac{R(E - I_2R_2)}{I_2(R + R_1)}$ R_2

$R = 25$ ohms, $R_1 = 5$ ohms, $E = 110$ v, $I_2 = 10$ amps

CHAPTER 11

Functions and Graphs of Functions

The freezing point of an automobile coolant depends on its density or specific gravity Distance traveled depends on velocity and time The stress developed in a material is related to the applied force Likewise, the volume of a sphere depends on the length of its radius

These statements serve to illustrate the concept of *functions*, a condition under which the value of one term or item, usually some variable, depends on or is controlled by another

11-1 FUNCTIONS

In mathematics and technology, when the value or behavior of one term controls the value or behavior of another, the second factor is referred to as a *function* of the first

The formula $C = \pi d$ means that the circumference of a circle, C, is a function of its diameter, d In functional notation, this is written as

$$C = f(d) = \pi d$$

The distance that a body falls depends on the time of flight

$$S = \frac{1}{2}gt^2 \quad \text{or} \quad f(t) = \frac{1}{2}gt^2$$

where t is called the *independent variable* and S is the *dependent variable*

In the equation $y = 2x - 3$, x is considered the independent variable and y the dependent variable In the equivalent equation $x = (y + 3)/2$, however, y is now the independent variable with x is the dependent variable

Similarly,

$$C = \pi d \text{ can be re-written as } d = \frac{C}{\pi}$$

where $f(C) = C/\pi$

The condition of the variable can change from independent to dependent. It is only a matter of position or placement of the term in the equation or formula.

Using the equation $y = x^2 - 3x + 1$ and changing it to functional notation results in this statement.

$$y = f(x) = x^2 - 3x + 1,$$

where the symbol $f(x)$ translates into the phrase, *y is a function of x* or *y is f of x*, which really means *that y depends on x for its value*. The symbol $f(x)$ does not affiliate with the *process* $f \cdot (x)$. The symbol $f(x)$, $f(y)$, $f(a)$, . . . , defines the following algebraic operation:

For $y = f(x) = x^2 - 3x + 1$, the value of $f(x)$ or y is determined by substituting various values of x, the independent variable, wherever x appears in the expression and performing the indicated algebraic manipulation:

$$f(x) = x^2 - 3x + 1$$

Then,

$$f(-2) = (-2)^2 - 3(-2) + 1 = 4 + 6 + 1 = 11$$

which states that for

$$x = -2, y \text{ or } f(x) = 11$$

further,

$$f(0) = (0)^2 - 3(0) + 1 = 1 \; (x = 0, y = 1)$$

and,

$$f(3) = (3)^2 - 3(3) + 1 = 9 - 9 + 1 = 1$$

The functional concept, $f(x)$, is used considerably in various branches of mathematics. This principle can apply to most any mathematical or engineering relationship involving several related terms or factors. It can be represented by an equation, formula, or by a graph-curve (geometrically). Recall that the concept of functions has already been incorporated in dimensional analysis (Chap. 5).

EXERCISES 11-1

Re-write the given expressions such that y becomes the independent variable, rather than x, as is now indicated. Final expressions should appear in functional notation ($y = 2x$, $f(y) = y/2$).

1. $y = 3x - 5$
2. $f(x) = 7 - 5x$
3. $y = \dfrac{2}{x} + 1$
4. $f(x) = \dfrac{1}{x - 1}$
5. $y = 4(x - 6) + 2x$
6. $f(x) = \dfrac{x^2 - 1}{1 + x}$

Find the value of the function as indicated

7. $f(x) = 3x - 10$, $f(1)$, $f(0)$, $f(-4)$

8. $f(x) = \dfrac{1}{2x - 5}$, $f(0)$, $f(3)$, $f(-2)$

9. $f(y) = 3y^2 - 2y + 1$, $f(0)$, $f(-3)$, $f(2)$

10. $f(r) = 2r - \dfrac{1}{2r}$, $f\left(\dfrac{1}{2}\right)$, $f(-1)$, $f(5)$

11. $f(x) = 7x^2 - 6x - 1$, $f(-3)$, $f(0)$, $f(a)$

12. $f(x) = \sqrt{x^2 - 9}$, $f(-3)$, $f(1)$, $f(a)$

13. $f(y) = \dfrac{y^2 - 1}{y^2 + 1}$, $f(1)$, $f(-1)$, $f(5)$

14. $f(x) = x^3 - 8x^2 - 5x + 3$, $f(0)$, $f(2)$, $f(-3)$

15. $f(t) = \dfrac{t^2 - 2t + 1}{t^2 - 1}$, $f(5)$, $f(-5)$, $f(10)$

11-2 RECTANGULAR COORDINATES

A brief mention of the *rectangular coordinate system*, or *Cartesian coordinate system*, was made in Sec 1-5 This discussion will continue from that point

In Fig 11-1, the units along the x-axis (horizontal) as well as the units along the y-axis (vertical) are laid off to the same arbitrary convenient scale

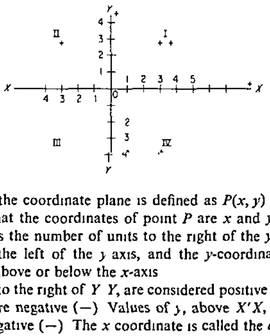


Figure 11-1

Every point in the coordinate plane is defined as $P(x, y)$ Translated, this symbol means that the coordinates of point P are x and y, where the coordinate x indicates the number of units to the right of the y-axis or the number of units to the left of the y axis, and the y-coordinate indicates the number of units above or below the x-axis

All values of x, to the right of Y Y, are considered positive (+), whereas those to the left are negative (−) Values of y, above $X'X$, are positive (+), those below, negative (−) The x coordinate is called the *abscissa* and the y-coordinate is referred to as the *ordinate* The system is further divided into four quadrants, as shown in Fig 11-1

All values of the abscissa and ordinate are positive in the first quadrant, whereas both are negative in the third quadrant. In quadrant II, x is negative and y is positive, whereas in the fourth quadrant, x, the abscissa, is $(+)$ and y the ordinate is $(-)$.

EXAMPLE 11-A:

Locate or plot the following points (Fig. 11-2).

$$P_1(4, 3),\ P_2(-5, 7),\ P_3(-3, -5),\ \text{and}\ P_4(6, -9)$$

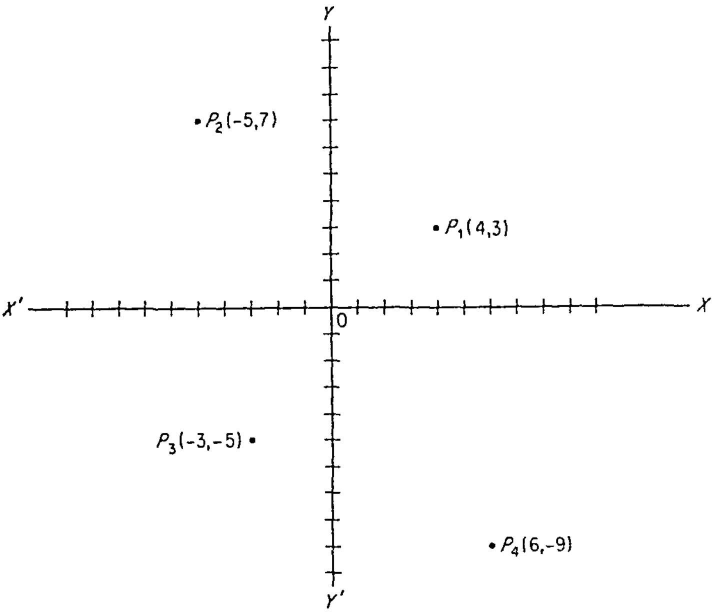


Figure 11-2

Solution:

Lay out a convenient rectangular coordinate system as in Fig. 11-2. $P(4, 3)$ indicates that $x = 4$ and $y = 3$, which means that the point lies 4 units to the right of the y-axis and 3 units above the x-axis. This location is then recorded with a dot $(\cdot)$ or small circle $(\circ)$ and labeled as $P(4, 3)$, or just $(4, 3)$. The subscript can be carried or dropped since the coordinates identify unique points. The subscripts are usually carried when the points are defined only with unassigned values of x and y, such as $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, . . . , $P_n(x_n, y_n)$. $P_2(-5, 7)$ is located 5 units to the left of the y-axis and 7 units above the x-axis and plotted accordingly. $P_3(-3, -5)$ defines a point 3 units to the left of the y-axis and 5 units below the x-axis. $P_4(6, -9)$ is a point 6 units to the right of the y-axis and 9 units below the x-axis.

The rectangular coordinate system is made up of a set of perpendicular axes that contain equal or identical units. In technology and business, a coordinate system may be used with an axis extending from an origin to the right and upward only. This would be equivalent to the first quadrant of the rectangular system, in which both variables are considered positive. Furthermore, the units laid off on the respective axes are invariably dissimilar,

although related voltage plotted against current, stress against force, temperature vs pressure, and so on

Graphs are *geometric representations* of some form of data or functional relationship They provide a visual study of various forms of information The data or points that are used to plot the curve or graph may be obtained from an equation, formula, experiment, statistics, or a series of laboratory tests

In engineering and mathematics, a smooth curve is desirable whereas statistical and experimental data may lead to a broken line A *smooth curve* represents a figure that does not take on any sudden breaks or departures from the general trend It may include circles, parabolas, straight lines, and any other form that can be described without the appearance of an abrupt shift (Fig 11-3)

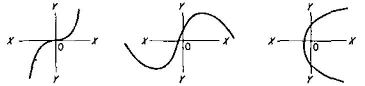


Figure 11-3

The initial study of graphs will involve smooth curves and straight lines A straight line can be referred to as a curve that does not change direction

11-3 PLOTTING FUNCTIONS

The *graph* of a function is the *locus* (path) of all points whose coordinates satisfy an equation or other functional relationships The points that are used in *plotting* or *graphing* a curve are obtained from the functional relationship, $y = f(x)$ Arbitrary values of x are substituted in $f(x)$ to find the corresponding values of y For some curves several sets of coordinates will be needed to complete the graph, whereas a straight line can be defined by two points only In the case of the straight line, however, three well-spaced points should be plotted as a measure of accuracy

Practice and experience will lead to the development of a technique whereby the general trend of a curve will be recognized with a minimum number of points For example, the degree of an equation will indicate, by and large, the type of a curve the function represents

> $f(x) = 3x - 2$, the exponent of the variable, x, is 1, which means that the function is *linear*, thus representing a *straight line*, $f(x) = ax + c$, $f(x) = (x^2 - 2)/16$, the power of the exponent is 2, which defines a quadratic A *second-degree equation* will appear as a *curve that changes direction once* (Fig 11-4)

The equation $f(x) = 3x^3 - 16$ is a *cubic* and the curve associated with this function will *change direction twice* (Fig 11-5) These and other character-

istics, such as maximum and minimum points, once recognized, enable the function to be sketched quickly.

In plotting or graphing a function, $f(x)$, an appropriate coordinate system is needed, usually suggested by the conditions of the problem. Along with this, some format of recording the points is required, as suggested in Fig. 11-6.

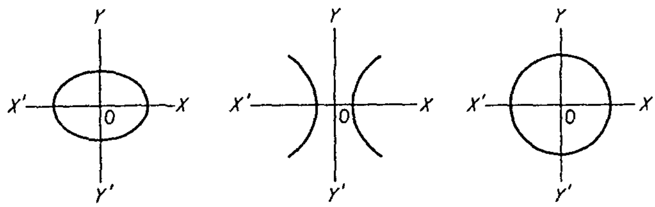


Figure 11-4

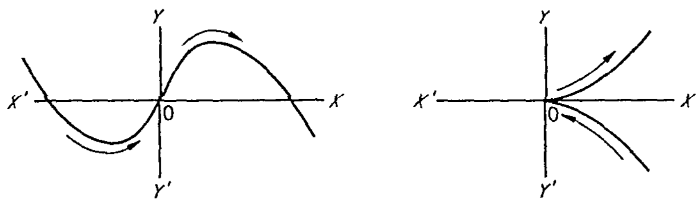


Figure 11-5

x	0	-2	3
$y = f(x)$	3	14	-6

or

x	$y = f(x)$
0	3
-2	14
3	-6

Figure 11-6

This is followed by substituting various values of x in the equation $y = f(x)$ to find the corresponding values of y. Thus, the coordinates of a point are determined.

It is advisable, at first, to choose values of the independent variable that are well spaced, especially if $f(x)$ is unfamiliar, such as $f(x) = 4x^4 - 3x^3 + 5x^2 - 6x + 7 = 0$. At times, however, it may become necessary to investigate consecutive integers in order to determine *maxima* and *minima.*

Maxima and minima are shortened expressions meaning *maximum* and *minimum.* A *maximum point* occurs where the function reaches a high point within a certain region, whereas a *minimum point* is a low point of the curve in an immediate area. Actually, at *maxima* the curve is concave downward whereas at *minima* the curve is concave upward (Fig. 11-7).

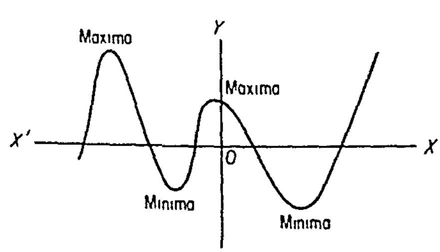


Figure 11-7

The final step involves plotting the points on a coordinate plane and constructing a smooth curve The practice of drawing a smooth curve through a series of plotted points is to line up the edge of the drawing instrument to coincide with three successive points The curved line is then drawn through the first two points only The procedure is repeated until all points are included in this manner (Fig 11-8a)

In the event that some points appear out of line with neighboring points, an average is taken to keep the concept of a smooth curve intact (Fig 11-8b)

(a) (b) **Figure 11-8**

EXAMPLE 11-B

Plot the function $y = 3x - 7$

Solution

Since $f(x)$ is linear, only three points will be required

$$y = f(x) = 3x - 7$$

Let $x = 0$, then $f(x) = f(0) = 3(0) - 7 = -7$

$x = 5$ $\quad f(5) = 3(5) - 7 = 15 - 7 = 8$

$x = -3$ $\quad f(-3) = 3(-3) - 7 = -9 - 7 = -16$

x	0	5	-3
$y = f(x)$	-7	8	-16

Locate the points on the coordinate plane and draw a line through them (Fig 11-9)

The points at which the straight line (or curve) crosses the axes are called *intercepts* *x-intercept* and *y-intercept*, respectively To find the *x-intercept*, set $y = 0$ in the equation and solve for x To find the *y intercept*, set $x = 0$ and solve for y In this example, the x-intercept is $(\frac{7}{3}, 0)$, whereas the y intercept is $(0, -7)$

The coordinates are well-spaced and it appears reasonally safe to assume that $y = 3x - 7$ is a straight line, as indicated

The same results should be obtained by expressing $f(x) = 3x - 7$ as a function of y, $f(y)$

$$y = 3x - 7, \text{ solving for } x$$

$$3x = y + 7, \text{ or } x = \frac{y + 7}{3}$$

$$f(y) = \frac{y+7}{3}$$

$$f(0) = \frac{0+7}{3} = \frac{7}{3}$$

$$f(-7) = \frac{-7+7}{3} = 0$$

$$f(5) = \frac{5+7}{3} = 4$$

y	$x = f(y)$
0	$\frac{7}{3}$
−7	0
5	4

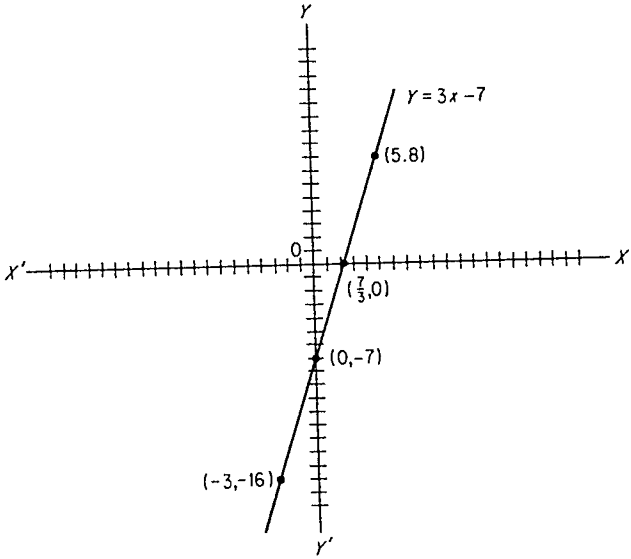


Figure 11-9

Plotting these points on the coordinate system of Fig. 11-9 leads to the same straight line (curve), $y = 3x - 7$, which of course was expected since $x = (y + 7)/3$ is an equivalent equation.

The procedure for plotting the function $f(x) = ax^2 + bx + c$ is basically the same as the preceding illustration. This function defines a parabola and it will be the only quadratic discussed. A parabola is the basis of design of various light and sound reflectors (Sec. 18-5b).

EXAMPLE 11-C:

Plot $y = f(x) = x^2/4$

By the nature of the equation $y = x^2/4$, the y-coordinate can never assume negative values, no matter what numbers are assigned to x.

Solution:

Set up a table, assign values to the independent variable, and solve for the corresponding value of the dependent variable.

$$y = f(x) = \frac{x^2}{4}$$

$x = 0,\ f(0) = \frac{0}{4} = 0$

$x = 2,\ f(2) = \frac{(2)^2}{4} = \frac{4}{4} = 1$

$x = -2,\ f(-2) = \frac{(-2)^2}{4} = \frac{4}{4} = 1$

$x = 4\ \ f(4) = \frac{(4)^2}{4} = \frac{16}{4} = 4$

$x = -4,\ f(-4) = \frac{(-4)^2}{4} = \frac{16}{4} = 4$

$x = 6,\ f(6) = \frac{(6)^2}{4} = \frac{36}{4} = 9$

$x = -6,\ f(-6) = \frac{(-6)^2}{4} = \frac{36}{4} = 9$

x	$y = f(x)$
0	0
2	1
−2	1
4	4
−4	4
6	9
−6	9

Perhaps for this particular function, enough points are presently available to establish the general form of the curve (Fig 11-10)

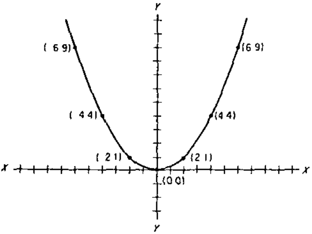


Figure 11-10

This happens to be a parabola with the vertex at the origin, opening upward The minimum point is at (0, 0) This curve does not have a maximum point

EXAMPLE 11-D

Graph the function $y = 4 - 3x - x^2$

Solution

Usually, after several random plots, a pattern may emerge, suggesting more meaningful selection of subsequent coordinates A common starting point is to set the independent variable equal to zero

$$f(x) = 4 - 3x - x^2$$
$$f(0) = 4 - 3(0) - (0)^2 = 4$$
$$f(2) = 4 - 3(2) - (2)^2 = 4 - 6 - 4 = -6$$
$$f(4) = 4 - 3(4) - (4)^2 = 4 - 12 - 16 = -24$$
$$f(-4) = 4 - 3(-4) - (-4)^2 = 4 + 12 - 16 = 0$$
$$f(-2) = 4 - 3(-2) - (-2)^2 = 4 + 6 - 4 = 6$$
$$f(1) = 4 - 3(1) - (1)^2 = 4 - 3 - 1 = 0$$

x	$y=f(x)$
0	4
2	−6
−2	6
4	−24
−4	0
1	0

These few scattered points give a sketchy trend of the parabola and suggest further study (Fig. 11-11a).

At this stage it appears that the parabola has reached its maximum somewhere between $x = 0$ and $x = -4$. This suggests that further trials include $x = -1$ and $x = -3$.

$$f(-1) = 4 - 3(-1) - (-1)^2 = 4 + 3 - 1 = 6$$
$$f(-3) = 4 - 3(-3) - (-3)^2 = 4 + 9 - 9 = 4$$

Adding these points to the preceding plot leads to Fig. 11-11b.

From the table and the incomplete graph, it now appears that the curve reaches its maximum at a point between $x = -2$, $(y = 6)$, and $x = -1$ $(y = 6)$. Trying $x = -3/2$ leads to:

$$f\left(\frac{-3}{2}\right) = 4 - 3\left(\frac{-3}{2}\right) - \left(-\frac{3}{2}\right)^2 = 4 + \frac{9}{2} - \frac{9}{4}$$
$$= \frac{16 + 18 - 9}{4} = \frac{25}{4}, \text{ where } \frac{25}{4} > 6 \text{ (maximum or maxima)}$$

Now that the maximum has been established, several more values of x will be taken to complete the left branch of the parabola.

$$f(-5) = 4 - 3(-5) - (-5)^2 = 4 + 15 - 25 = -6$$
$$f(-7) = 4 - 3(-7) - (-7)^2 = 4 + 21 - 49 = -24$$

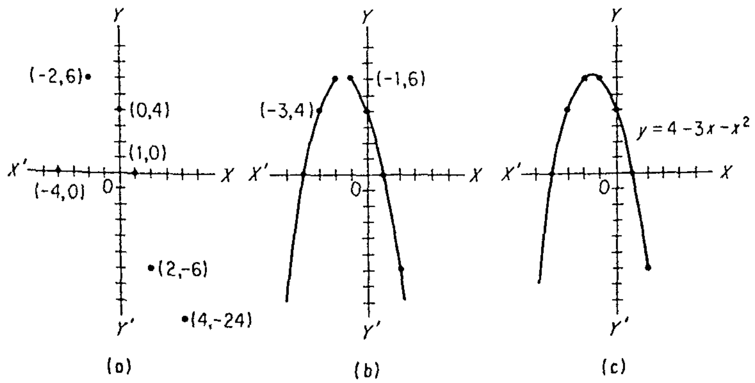


Figure 11-11

Enough points are now available to complete the approximation (Fig 11-11c) This curve has no minimum point

EXAMPLE 11 E:

Graph the function $y = x^3 - 6x^2 + 5$

Solution

The procedure remains basically the same as in the preceding example, assign values to x and solve for y With a cubic, naturally, there is more involvement and the curve is not a simple one

$$f(x) = x^3 - 6x^2 + 5 = 5$$
$$f(0) = (0)^3 - 6(0) + 5 = 5$$
$$f(2) = (2)^3 - 6(2)^2 + 5 = 8 - 24 + 5 = -11$$
$$f(-2) = (-2)^3 - 6(-2)^2 + 5 = -8 - 24 + 5 = -27$$
$$f(4) = (4)^3 - 6(4)^2 + 5 = 64 - 96 + 5 = -27$$
$$f(-4) = (-4)^3 - 6(-4)^2 + 5 = -64 - 96 + 5 = -155$$
$$f(6) = (6)^3 - 6(6)^2 + 5 = 216 - 216 + 5 = 5$$

x	$f(x)$
0	5
2	−11
−2	−27
4	−27
−4	−155
6	5
−1	−2
1	0
3	−22
5	−20

At this stage it will probably be well to stop and plot the few points and analyze the trend, if one is starting to develop (Fig 11-12a)

It would appear that the branches of the curve are in evidence Recall that the cubic changes direction twice, it would appear from the incomplete sketch that the general form of the curve is apparent

Further study of the function indicates that, after a certain range of x, the value of the function will be dominated by the term x^3, where for large values of x the other terms $(-6x^2 + 5)$ will have negligible effect on the direction that the function will assume

For

$$x = 100, \quad x^3 = (100)^3 = (10^2)^3 = 10^6 = 1{,}000{,}000$$
$$-6x^2 = -6(100)^2 = -6 \times 10^4 = -60{,}000$$

For

$$x = -100, \quad x^3 = (-100)^3 = -10^6 = -1{,}000{,}000$$
$$-6x^2 = -6(-100)^2 = -6 \times 10^4 = -60{,}000$$
$$1{,}000{,}000 - 60{,}000 = 940{,}000$$

which is a large number, influenced more by x^3 than by $-6x^2$

From a study of the table as well as the incomplete sketch, the behavior of the curve between $x = -2$ and $x = 6$ appears to be most critical.

Continuing with more coordinates,

$$f(-1) = (-1)^3 - 6(-1)^2 + 5 = -1 - 6 + 5 = -2$$
$$f(1) = (1)^3 - 6(1)^2 + 5 = 1 - 6 + 5 = 0$$

This would indicate that the curve reaches a maximum point when $x = 0$, since in the immediate area of the curve $f(-1) = -2$ and $f(1) = 0$, both of which are less than $f(0) = 5$. A maximum point is not always indicative of the maximum value of the function, but rather is simply a point where the curve is concave downward. It is a concept rather than a true quantitative measure (Fig. 11-12b).

Furthermore, it seems apparent that the curve reaches a minimum between $x = 1$ and $x = 6$.

$$f(3) = (3)^3 - 6(3)^2 + 5 = 27 - 54 + 5 = -22$$
$$f(5) = (5)^3 - 6(5)^2 + 5 = 125 - 150 + 5 = -20$$

Thus, it appears that the coordinates $(4, -27)$ define the point where the curve reaches a minimum or is concave upward;

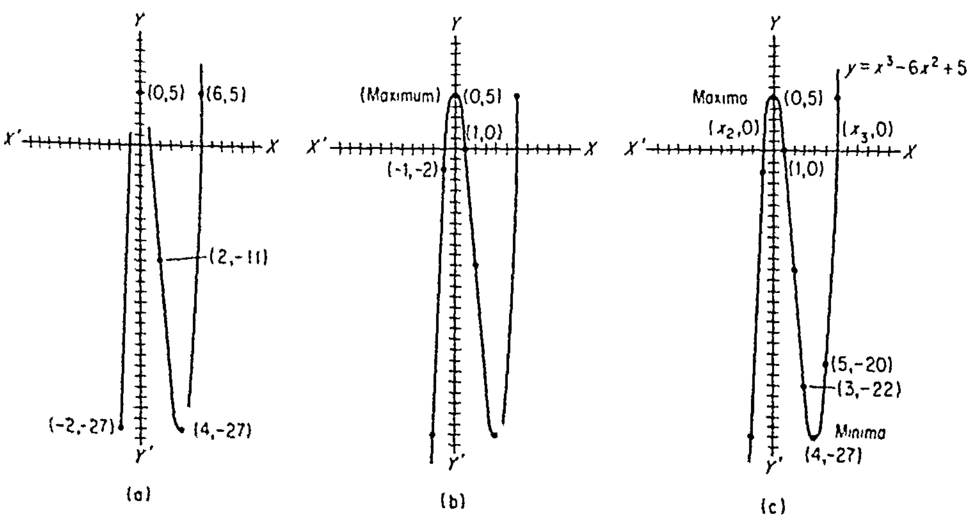
$$f(3) = -22,\ f(4) = -27, \text{ and } f(5) = -20$$

Along with this, two of the intercepts are known:

y-intercept $(0, 5)$ and x-intercept $(1, 0)$

The curve can now be constructed with some degree of accuracy (Fig. 11-12c).

Figure 11-12



The other two x-intercepts can be approximated

$$-1 < x_2 < 0, \text{ and } 5 < x_3 < 6$$

EXERCISES 11-2

1. Locate the following points on a coordinate plane and draw a smooth curve through them Identify the curve

$(5, 0)$, $(-4, -3)$, $(2, \sqrt{21})$, $(3, 4)$, $(0, -5)$, $(-4, 3)$, $(1, -2\sqrt{6})$, $(3, -4)$, $(-2, -\sqrt{21})$, $(-5, 0)$, $(-3, -4)$, $(-2, \sqrt{21})$, $(4, 3)$, $(-3, 4)$, $(1, 2\sqrt{6})$, $(0, 5)$, $(2, -\sqrt{21})$, $(4, -3)$, $(-1, -2\sqrt{6})$, $(-1, 2\sqrt{6})$ $(\sqrt{21} = 4\ 6, \sqrt{6} = 2\ 4)$

Plot the following straight lines and determine the intercepts (Ex 2-12)

2. $x = y$

3. $y = 3x - 1$

4. $2y = 3x - 1$

5. $6x - 7 = y$

6. $y = 15 - 9x$

7. $y = \frac{x}{2} - \frac{1}{5}$

8. $y = \frac{5x - 3}{4}$

9. $x = 2y - 10$

10. $3x + y - 10 = 0$

11. $5x - 15y = 10$

12. (a) $x = 0$, (b) $y = 0$

Graph the following functions (determine approximate maxima and minima)

13. $y = 16x^2$

14. $4y = -16x^2 - 8$

15. $y = 2x^2 - 3x + 5$

16. $y = \frac{1}{x^2 + 1}$

17. $y^2 = 6x^3$

18. $y = 4x^3 - x + 1$

19. $y = 4x^3 - x^2 + 1$

20. $y = \frac{1}{x^2}$

21. $xy = 1$

22. $y = \frac{1}{x^3}$

23. $y = (x - 2)(x + 2)$

24. $x^2 = y^2$

25. (a) $x - 5 = 0$, (b) $y + 5 = 0$

11-4 GRAPHS

There are many forms of graphs and this might very well be expected, since a graph is a visual representation involving related factors unique to a given illustration The technician will be mostly concerned with line graphs that, by and large, turn out as smooth curves

The information contained in the graph is usually obtained through experiments, laboratory tests, or other methods not defined by an equation or formula. Some graphs will contain a series of related topics, whereas others will plot the behavior or relationship of two items. Several such graphs will be illustrated (Fig. 11-13).

Statistical work concerning sales, daily temperature, or other information taken at specific time intervals usually is represented by a **broken line** (Fig. 11-14).

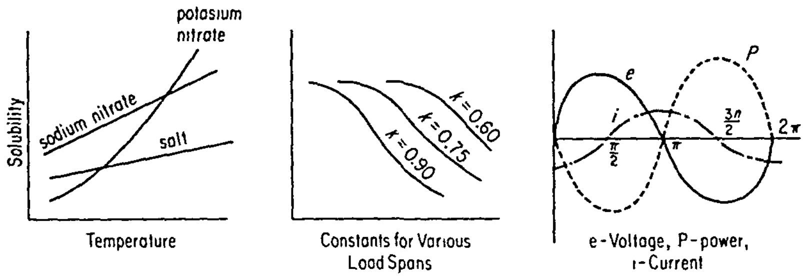


Figure 11-13

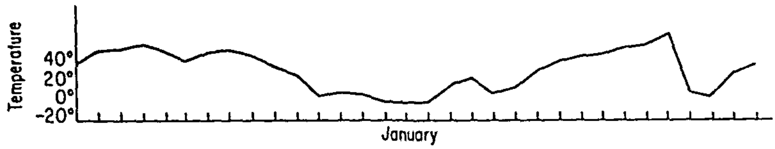


Figure 11-14

Line graphs are constructed in much the same manner as graphs of functions. A suitable scale is determined for the ordinates with a complimentary scale for the abscissa. Points are located accordingly, followed by an appropriate drawing.

A *bar graph* is an effective instrument to compare a statistic with several related components: the capacity of various reservoirs, the depth of adjoining lakes, production of grain in key states, age groups at the state university, and so on.

EXAMPLE 11-F:

Construct a bar graph showing the relationship of the leading cotton-producing nations.

Country	Production: 10^6 bales
United States	12.5
Russia	6.5
China	6.0
India	4.0
Mexico	2.0
Egypt	1.8
Brazil	1.5
Pakistan	1.4
Turkey	1.0

Construction

The horizontal scale contains the subject or the name of the element involved with the statistic This carries no specific unit of measure The vertical scale is representative of the measurement or quantitative statistic and is laid off to some suitable scale The top of the vertical bar reflects the measure of the item pertaining to the subject The width of the bar carries no meaningful data, except eye appeal

On the vertical scale, let 1 0 in $= 4 \times 10^6$ bales, the width of the bar will be taken as $\frac{1}{8}$ in The bars can be spaced or compressed (Fig 11-15)

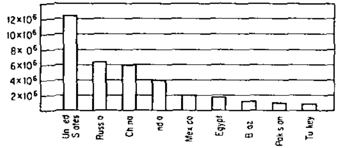


Figure 11-15

EXAMPLE 11 G

Construct a circle graph showing a family budget as indicated

Food	25%	Medical	10%
Utilities	5%	Housing	30%
Furniture	5%	Transportation	10%
Clothing	10%	Savings	5%
			100%

Construction

Determine the magnitude of the angle of each item (sector) by multiplying the respective percentages by 360°

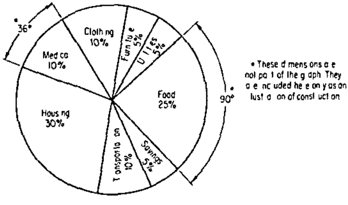


Figure 11-16

Food	$25\% \times 360° = 90°$
Utilities	$5\% \times 360° = 18°$
Furniture	$5\% \times 360° = 18°$

Clothing:	$10\% \times 360° = 36°$
Medical:	$10\% \times 360° = 36°$
Housing:	$30\% \times 360° = 108°$
Transportation:	$10\% \times 360° = 36°$
Savings:	$5\% \times 360° = 18°$

With compass and protractor, the construction is completed (Fig. 11-16).

EXERCISES 11-3

1. Draw a bar graph that shows the relationship of the leading copper mining states and provinces.

State—Province	Copper Production (tons)
Arizona	5.0×10^5
Utah	2.3×10^5
Ontario	1.6×10^5
Montana	1.0×10^5
Nevada	0.8×10^5
New Mexico	0.7×10^5
Michigan	0.6×10^5
Saskatchewan	0.3×10^5
British Columbia	0.2×10^5

2. Construct a circle graph giving a breakdown of the world-wide chemical industry.

Country	% World Production
Japan	5.0
Others	6.0
France	7.0
Italy	7.0
Great Britain	10.0
West Germany	10.0
United States	55.0
	100.0%

3. The breakdown of technician-type programs offered in the community colleges of a large midwestern state is as follows:

Programs	No. of Programs	% of Total
Agriculture	60	7.8
Business-Commerce	120	15.6
Data Processing	60	7.8
Health-Medical	100	13.0
Public-Social Service	80	10.3
Secretarial	110	14.3

Technology	175	22 7
Trades-Crafts	50	6 5
Others	15	2 0
	770	100 0%

Construct a bar graph and a circle graph, respectively, to represent the given information *Recall* that graphs are at best approximations

4. Draw a graph of a resistance-inductance (*RI*) circuit given the following data

Current (ordinate) (amps)	Time (abscissa) (seconds)
0 00	0 00
0 20	0 02
0 40	0 04
0 50	0 06
0 60	0 08
0 65	0 10
0.70	0 12
0 73	0 14
0 74	0 18
0 75	0 20

5. A laboratory test was conducted in which axial loads were applied to a steel column Plot a curve from the data obtained

Stress (pounds per square inch)	Slenderness ratio (inches/inches)
16,000	0
16,000	10
16,000	20
16,000	30
16,000	40
16,000	50
15,500	60
15,000	70
14,500	80
14,000	90
13,500	100

The stress referred to in the data indicates the maximum stress that a column can be subjected to for a corresponding slenderness ratio This means that for a given cross-section, as the length of a column increases, the safe load decreases

6. The curve of the axis of a beam subjected to various loads is called the elastic curve of the beam A beam 20 0 ft long, fixed at both ends, bears a 12,000-lb load 5 0 ft from the right support (Fig 11-17) Deflection readings

were taken at 1.0-ft intervals, starting from the left support. The data is recorded below. Plot the elastic curve for the given conditions.

Distance from left support, d, in feet; deflection, Δ, in inches.

d	Δ	d	Δ
0	0	11.0	0.93
1.0	0.10	12.0	0.96
2.0	0.20	13.0	1.00
3.0	0.31	14.0	0.96
4.0	0.42	15.0	0.92
5.0	0.53	16.0	0.85
6.0	0.60	17.0	0.70
7.0	0.70	18.0	0.52
8.0	0.80	19.0	0.30
9.0	0.85	20.0	0.00
10.0	0.90		

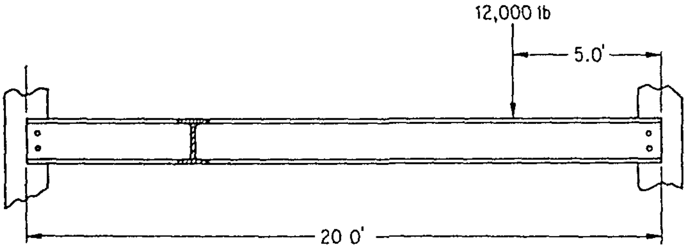


Figure 11-17

7. The data below gives the effect of temperature on a volume of gas when the pressure was held constant. Plot a curve using this information. Let the temperature be represented as the ordinate.

Temperature (°C)	Volume (milliliters)
0°	100.0
20°	107.4
40°	114.8
60°	122.2
80°	129.6
100°	137.0

REVIEW EXERCISES 11-4

Plot the given functions and determine intercepts whenever possible. Where applicable, approximate maximum and minimum points.

1. $2x = \frac{y}{3}$
2. $2x - 5y = 10$
3. $f(x) = 4x^2 - 5x$
4. $f(x) = 6 - 5x + x^2$
5. $y = \frac{1}{x^4}$
6. $3x = 0$

48,000	0.0110
50,000	0.0150
51,750	0.0210
50,000	0.0270
48,000	0.0310
46,000	0.0400

CHAPTER 12

Systems of Linear Equations

In mathematics as well as technology, there are certain relationships that cannot be defined completely by one equation alone Conditions involving more than one equation are referred to as a *system of equations* or *simultaneous equations*

The discussion of simultaneous equations will be limited, primarily, to two linear equations (straight lines) in two variables The procedures about to be established concerning these limited conditions, however, are applicable to other systems Basically, there are three methods of solving a set of related equations **graphically, algebraic addition or subtraction,** and **algebraic substitution** Except for a few special cases, the system is considered *solved* when a set of coordinates are determined that satisfy both equations This solution can be interpreted geometrically as the point of intersection of the two lines defined by the given equations

12-1 GRAPHICAL SOLUTION

The graphical approach to the solution of simultaneous equations involves plotting the functions and approximating the point of intersection, if one exists The coordinates of the point of intersection will represent the solution of the given system Since this method is really an approximate solution, it is used more as a geometric check of an algebraic solution rather than as the primary approach

EXAMPLE 12-A.

Solve the given system of linear equations, graphically

$$3x - y = 3$$
$$3x + 2y = 12$$

Solution:

The given equations are both linear and as such represent two straight

lines. A straight line can be plotted if two distinct points are known. Usually the intercepts serve this purpose conveniently.

For $3x - y = 3$, the intercepts are $(0, -3)$ and $(1,0)$.

For $3x + 2y = 12$, the intercepts are $(0, 6)$ and $(4, 0)$.

Recall that the y-intercept is determined by setting $x = 0$ and solving the remaining equation for y. Similarly, the x-intercept is found by setting $y = 0$ and solving for x. Figure 12-1 represents the system.

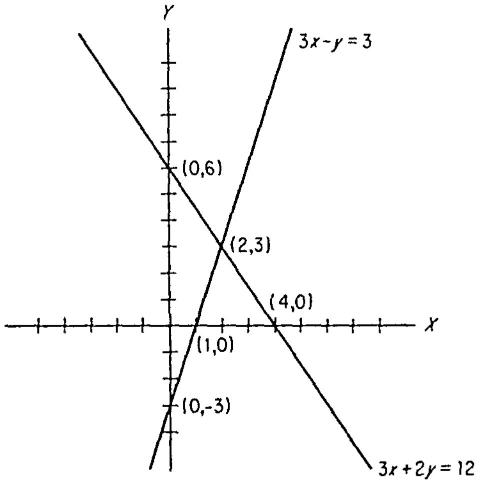


Figure 12-1

From the graphical representation in Fig. 12-1, it appears that the solution is (2, 3). This can be verified by substituting the coordinates (2, 3) into the given equations. If the equations are satisfied, the system is said to be solved. It should be pointed out that since this is an approximate solution, the check must be accepted with the same degree of accuracy.

Checking:

The coordinates (2, 3) are substituted into the given equations accordingly.

$$\begin{aligned} 3x + 2y &= 12 & \qquad\qquad 3x - y &= 3 \\ 3(2) + 2(3) &= 12 \quad \text{and} & 3(2) - (3) &= 3 \\ 6 + 6 &= 12 & 6 - 3 &= 3 \end{aligned}$$

which indicates that the equations balance. Thus (2, 3) is the solution of the given simultaneous equations.

EXAMPLE 12-B:

Solve the given system of equations graphically.

$$x - 2y = 4$$
$$3x - 6y = 24$$

Solution:

Find the intercepts and plot the lines (Fig. 12-2).

The intercepts of $x - 2y = 4$ are (4, 0) and (0, −2)
The intercepts of $3x - 6y = 24$ are (8, 0) and (0, −4)

From the graphical representation (Fig 12-2), it appears that the lines do not intersect Hence, there is no solution The two lines of discussion are parallel and when this condition exists, the system is called *inconsistent*

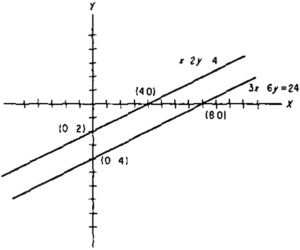


Figure 12-2

Parallel lines can be identified by inspection If the coefficients of the variables, taken respectively, are proportional, the lines are either parallel or coincident

Note

$$x - 2y = 4 \qquad \frac{1}{3} = \frac{-2}{-6}$$
$$3x - 6y = 24$$

EXAMPLE 12 C

Solve, graphically, the following pair of equations

$$5x + 2y = 10$$
$$35x + 14y = 70$$

Solution

Determine intercepts and plot the functions (Fig 12-3)
The intercepts of $5x + 2y = 10$ are (2, 0) and (0, 5)
The intercepts of $35x + 14y = 70$ are (2, 0) and (0, 5)

Since the intercepts are identical, both equations define the same line These equations are termed *dependent* equations Furthermore, this system has an unlimited number of solutions Every point on one line also lies on the other line

Coincident lines can also be identified by inspection Here, the coefficients of the variables, and the constant terms, taken respectively and in the same order, have the same ratio

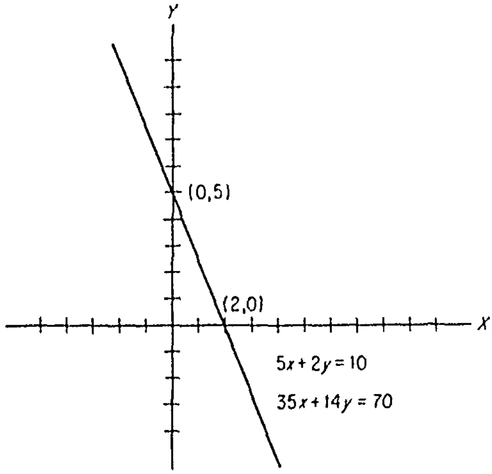


Figure 12-3

Note:

$$5x + 2y = 10 \qquad \frac{5}{35} = \frac{2}{14} = \frac{10}{70} = \frac{1}{7}$$
$$35x + 14y = 70$$

EXERCISES 12-1

Solve the following systems of linear equations graphically. Indicate, if there is no solution, whether the pair of equations is dependent or inconsistent (estimate to 0.25 unit).

1. $x + 2y = 8$
$2x + y = 8$

2. $2x - 3y + 16 = 0$
$x + y - 2 = 0$

3. $5x + 3y = 15$
$5x - 3y = 0$

4. $4x - 3y = 12$
$3x + 4y = 12$

5. $7x + 3y = 21$
$28x + 12y = 63$

6. $6x - 5y = 21$
$2x + 3y = 7$

7. $3.5x - 7.5y = 0$
$6.0x + 4.5y = 0$

8. $2x + y = 1$
$6x - 5y = 15$

9. $2y + 4x + 7 = 0$
$y - x - 1 = 0$

10. $10x - 14y = 22$
$\frac{5}{3}x - \frac{7}{3}y = \frac{11}{3}$

12-2 SOLUTION BY ADDITION OR SUBTRACTION

Most often, simultaneous equations are solved by algebraic methods. Graphical solutions are approximate, whereas algebraic solutions are considered precise. One approach to an algebraic solution of two equations in two variables is called the *addition or subtraction* method. This method is utilized when the coefficients of one of the variables are equal in absolute value.

The principle of eliminating one variable leads to an expression defining the other variable This quantity is then substituted into either of the original equations to determine the remaining unknown Several examples will demonstrate the process associated with the solution of a system of linear equations by the method of addition or subtraction

EXAMPLE 12 D

Solve the given system of linear equations by the method of addition or subtraction

$$3x - 4y = -7$$
$$5x + 4y = 31$$

Solution

If the coefficients of one of the variables are equal, including signs, elimination is accomplished by subtraction If the coefficients are equal but opposite in sign, addition is used In this example, the equations will be added Adding

$$\begin{array}{rl} 3x - 4y & = -7 \\ \underline{5x + 4y} & \underline{= 31} \\ 8x \qquad & = 24, \text{ from which } x = 3 \end{array}$$

The value of y can be determined by substituting 3 for x in one of the given equations

In the equation $3x - 4y = -7$, let $x = 3$

$$3(3) - 4y = -7$$

Collecting terms,

$$-4y = -16 \quad \text{and} \quad y = 4$$

Thus, the solution of the given system appears to be $x = 3$ and $y = 4$ This solution can be checked or verified by substituting back into the second equation, $5x + 4y = 31$

$$(3, 4) \quad 5(3) + 4(4) = 31$$
$$15 + \quad 16 = 31$$

which indicates that the equation is satisfied and (3, 4) is the solution of the given system

EXAMPLE 12 E

Solve the given system of equations

$$\frac{3}{2}x - \frac{5}{2}y + 39 = 0$$
$$\frac{5}{3}x + \frac{7}{3}y - 18 = 0$$

Solution:

Whenever equations appear in fractional form, it is suggested that they be simplified by multiplying through by the least common denominator.

Multiplying by 2: $\frac{3}{2}x - \frac{5}{2}y = -39$ becomes $3x - 5y = -78$

Multiplying by 3: $\frac{5}{3}x + \frac{7}{3}y = 18$ becomes $5x + 7y = 54$

The method of addition or subtraction requires that a pair of coefficients be equal. In some instances this will involve multiplication by appropriate factors to convert the given equations into suitable forms, such as multiplying $3x - 5y = -78$ by 5 and $5x + 7y = 54$ by 3. This leads to:

$$\begin{array}{lrl} & 15x - 25y = & -390 \\ & 15x + 21y = & 162 \\ \hline \text{Subtracting:} & -46y = & -552 \text{ where } y = 12 \end{array}$$

Next, substituting $y = 12$ into one of the given equations leads to the solution of x.

$$3x - 5y = -78$$

$$y = 12; \qquad 3x - 5(12) = -78,$$

which when simplified becomes

$$3x = -18 \quad \text{and} \quad x = -6$$

Thus, the apparent solution to the given system is $x = -6$ and $y = 12$, or $(-6, 12)$.

Checking:

Substituting $(-6, 12)$ into the equation $\frac{5}{3}x + \frac{7}{3}y = 18$ leads to:

$$\frac{5}{3}(-6) + \frac{7}{3}(12) = 18$$

$$-10 + \quad 28 = 18$$

which confirms the solution.

12-3 SOLUTION BY SUBSTITUTION

The method of solution by *substitution* involves solving one of the given equations in terms of the other members. This quantity is then substituted for the equivalent variable in the second equation. The resulting expression will lead to the solution of one of the unknowns. The remaining variable is determined as previously explained.

EXAMPLE 12-F:

Solve the given system of linear equations by the method of substitution.

$$3x - y = 17$$
$$5x + 3y = 19$$

Solution

This method is most convenient when one of the coefficients is equal to 1 or -1 In any event, if the choice of solution is by the method of substitution, the variable with the simplest coefficient should be considered Thus, the equation $3x - y = 17$ will be solved for in terms of y, where $y = 3x - 17$

Substituting for the equivalent variable in the second equation leads to the following expression

$$5x + 3(3x - 17) = 19$$

Simplifying

$$5x + 9x - 51 = 19$$
$$14x = 70 \quad \text{and} \quad x = 5$$

Substituting $x = 5$ into the equation $y = 3x - 17$ will lead to the solution of y

$$y = 3(5) - 17 = 15 - 17 = -2$$

Therefore, the solution to the given system is $x = 5$ and $y = -2$, or $(5, -2)$ The solution can be checked by substituting the coordinates, $5, -2$, into the equation $5x + 3y = 19$ and simplifying

EXERCISES 12-2

Solve each of the systems of equations by the method that appears to be most convenient Check graphically or algebraically, whichever seems simplest

1. $2x + y = 4$
$3x - y = 6$

2. $5x + 2y = 10$
$5x + 3y = 5$

3. $3x + 6y = 0$
$5x - 5y = 0$

4. $3x + 6y - 12 = 0$
$2x - 7y + 25 = 0$

5. $4x + 5y + 19 = 0$
$3x - 4y - 40 = 0$

6. $8x - 9y = 30$
$4x + 7y = -54$

7. $3x = 6(y - 3)$
$3y = 6(x - 6)$

8. $11x + 13y + 3 = 0$
$5x - 3y + 37 = 0$

9. $2x + 3y = -6$
$8x + 5y = -3$

10. $x + 2y + 3 = 0$
$6x - 9y - 31 = 0$

11. $\frac{x}{4} + \frac{y}{5} = 0$
$\frac{3}{4}x - \frac{3}{5}y = 12$

12. $1.25x - 3.50y + 14.50 = 0$
$3.75x + 6.50y + 9.50 = 0$

13. $3x + 5y = 3x - 5y + 10$
$6y - 5x = 6$

14. $7x - 4y + 1 = 5x - 4y - 5$
$3y = 3(x - 2)$

15. $6x + 4y = 25$
$7x - 3y = 56$

16. $\sqrt{3}x + \sqrt{2}y = 5$
$5\sqrt{3}x - 4\sqrt{2}y = 7$

17. $2\sqrt{5}x - \sqrt{7}y = 17$
$5\sqrt{5}x + 7\sqrt{7}y = -24$

18. $\frac{3}{x} + \frac{4}{y} = 1$
$\frac{6}{x} - \frac{8}{y} = 0$

19. $\frac{1}{2x} + \frac{1}{3y} = 2$
$\frac{1}{4x} + \frac{1}{6y} = 1$

20. $3ax + 7ay = 5a$
$4ax + 5ay = -2a$

21. $4x + 3y = 3a$
$5x + 4y = 5a$

22. $3x - y = \sqrt{10}$
$13x - 6y = \sqrt{10}$

Solve each of the following systems graphically (estimate to the nearest 0.5 unit).

23. $x = y$
$xy = 1$

24. $y^2 = 4x$
$2x - y = 4$

25. $y^2 = -16x$
$x^2 = 4y$

26. $x^2 + y^2 = 25$
$2x - y = 5$

CHAPTER 13

Quadratic Equations

An equation of the type $2x^2 - 3x + 7 = 0$, $4x^2 - 8x = 0$, or $5y^2 - 16 = 0$ is called a *quadratic equation* A quadratic equation is also referred to as a second degree equation since the highest power of the exponent is 2

The standard form of a quadratic equation appears as

$$y = f(x) = ax^2 + bx + c = 0$$

If $b = 0$, the quadratic takes on the form $ax^2 + c = 0$ and is called an *incomplete quadratic* or pure *quadratic equation* $(5y^2 - 16 = 0)$ If $c = 0$, the equation becomes $ax^2 + bx = 0$ and this is also known as an *incomplete quadratic equation* $(4x^2 - 8x = 0)$

13-1 SOLUTION BY FACTORING

The principle methods of solving quadratic equations are *plotting*, *factoring*, *completing the square*, and applying the *quadratic formula* Plotting provides an approximation of roots and is used primarily to check an algebraic solution or a graphical representation

If $ax^2 + bx + c = 0$ can be factored, where $ax^2 + bx + c = (px + r)(qx + s) = 0$, each factor is then set equal to zero and solved in terms of an algebraic equation

$$\text{If} \quad px + r = 0, x = -\frac{r}{p}$$

$$\text{If} \quad qx + s = 0, x = -\frac{s}{q}$$

This method of solving a quadratic equation is based on the following concept

The product of two or more factors is equal to zero if and only if one or more of the factors is equal to zero

EXAMPLE 13-A:

Solve the equation $4x^2 - 8x = 0$, using the factoring method.

Solution:

Factoring: $4x^2 - 8x = 0$ becomes $4x(x-2) = 0$
Now, $4x(x-2) = 0$, if $4x = 0$ or $x - 2 = 0$

Thus, setting each factor equal to zero leads to the solution of the equation.

$$4x = 0 \quad \text{and} \quad x = 0$$
$$x - 2 = 0 \quad \text{and} \quad x = 2$$

Hence, the solution of $4x^2 - 8x = 0$ appears to be $x = 0$ and $x = 2$. These are also referred to as the roots of the given equation. Geometrically, the roots can be represented as the x-intercepts, the points where the curve crosses the x-axis (Fig. 13-1).

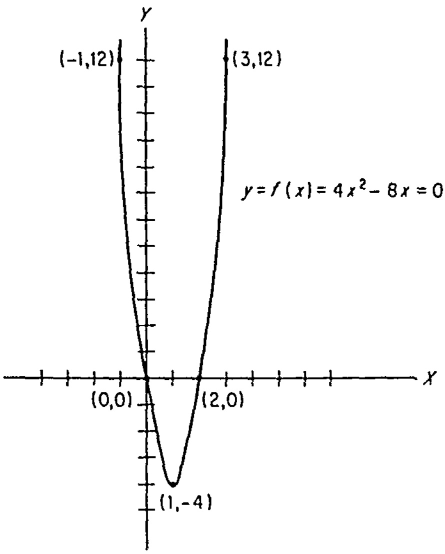


Figure 13-1

Checking:

If the roots satisfy the given equation, the solution is correct.

$$x = 0,\ 4x^2 - 8x = 4(0) - 8(0) = 0$$
$$x = 2,\ 4(2)^2 - 8(2) = 16 - 16 = 0$$

The equation is satisfied in both cases; thus, $x = 0$ and $x = 2$ are the roots of $4x^2 - 8x = 0$.

EXAMPLE 13-B:

Solve $2x^2 + 5x - 12 = 0$ by the factoring method. Check graphically.

Solution

Factor the given trinomial, set factors equal to zero, and solve for x

$$2x^2 + 5x - 12 = 0$$
$$(2x - 3)(x + 4) = 0$$
$$2x - 3 = 0 \quad \text{and} \quad x = \frac{3}{2}$$
$$x + 4 = 0 \quad \text{and} \quad x = -4$$

Thus, the roots appear to be $x = \frac{3}{2}$ and $x = -4$

Checking (graphically, Fig 13-2)

$$y = f(x) = 2x^2 + 5x - 12 = 0$$
$$f(0) = 2(0)^2 + 5(0) - 12 = 0 + 0 - 12 = -12$$
$$f(2) = 2(2)^2 + 5(2) - 12 = 8 + 10 - 12 = 6$$
$$f(-2) = 2(-2)^2 + 5(-2) - 12 = 8 - 10 - 12 = -14$$
$$f(4) = 2(4)^2 + 5(4) - 12 = 32 + 20 - 12 = 40$$
$$f(-3) = 2(-3)^2 + 5(-3) - 12 = 18 - 15 - 12 = -9$$
$$f(-5) = 2(-5)^2 + 5(-5) - 12 = 50 - 25 - 12 = 13$$

x	$y = f(x)$
0	-12
2	6
-2	-14
4	40
-3	-9
-5	13

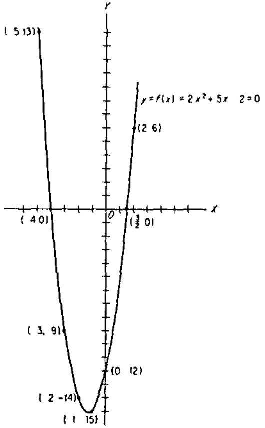


Figure 13-2

These points seem to provide enough information to sketch the parabola.

From the graph, it appears that the roots are $x = \frac{3}{2}$ and $x = -4$

EXAMPLE 13-C:

Solve the equation $4x^2 - 25 = 0$.

Solution:

Transpose and solve for x.

$$4x^2 - 25 = 0$$
$$4x^2 = 25$$
$$x^2 = \frac{25}{4}$$

Extracting square roots from both sides leads to:

$$x = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$$

where

$$x = \frac{5}{2} \quad \text{and} \quad x = -\frac{5}{2}$$

The equation can also be solved by factoring.

$$4x^2 - 25 = 0 \text{ is the difference of two squares}$$

Thus;

$$4x^2 - 25 = (2x - 5)(2x + 5) = 0$$

Hence,

$$2x - 5 = 0 \quad \text{and} \quad x = \frac{5}{2}$$
$$2x + 5 = 0 \quad \text{and} \quad x = -\frac{5}{2}$$

EXERCISES 13-1

Solve the following quadratics by the factoring method. Check even-numbered exercises graphically and odd-numbered exercises algebraically.

1. $6x^2 - 42x = 0$
2. $3x - 15x^2 = 0$
3. $6x^2 - 216 = 0$
4. $x^2 + 7x = 0$
5. $3x^2 = 8x$
6. $3x^2 + 8x = 0$
7. $3x^2 - 27 = 0$
8. $x^2 - 7x - 8 = 0$

9. $3y^2 - 6y + 3 = 0$

10. $4y^2 + 20y + 25 = 0$

11. $r^2 - 18r + 81 = 0$

12. $10t^2 + 26t - 56 = 0$

13. $\frac{x}{3} = \frac{9}{x}$

14. $4x^2 = 64$

15. $4y^2 = 8a^2$

16. $4x^2 - 8x + 13 = 3x^2 - 3x + 19$

17. $3x(x + 1) = 6$

18. $(x - 4)(x + 4) = 6x - 25$

19. $(17x - 4)(5x + 2) = 5x + 2$

20. $3x^2 - \frac{3}{4} = 0$

13-2 SOLUTION BY COMPLETING THE SQUARE

Not all quadratic equations are subject to factoring At times conditions are such that other methods must be employed to extract a solution One of these methods is known as *completing the square*

Fundamentally, this method is developed around the concept of converting a given quadratic into a perfect square The procedure will be illustrated through the solution of a problem

EXAMPLE 13-D

Solve the equation $x^2 - 6x - 16 = 0$ by the method of completing the square

Solution

The initial step involves isolating the variables on one side of the equation and the constant term on the other side This can be accomplished by transposing the constant term to the right

$$x^2 - 6x - 16 = 0$$

becomes

$$x^2 - 6x = 16$$

Next, divide the coefficient of the linear term by 2 and add the square of the quotient to both sides

$$\left(-\frac{6}{2}\right)^2 = (-3)^2 = 9$$

$$x^2 - 6x + 9 = 9 + 16 = 25$$

This step converts the given equation into an equivalent equation containing a perfect square trinomial Thus, $x^2 - 6x - 16 = 0$ is equivalent to $x^2 - 6x + 9 = 25$, where $x^2 - 6x + 9$ is a perfect square trinomial

$$x^2 - 6x + 9 = (x - 3)^2 \quad \text{Hence, } (x - 3)^2 = 25$$

Extracting square roots and solving for x leads to completion of the problem.

$$x - 3 = \pm\sqrt{25} = \pm 5$$

Thus,

$$x - 3 = 5 \quad \text{and} \quad x = 8$$
$$x - 3 = -5 \quad \text{and} \quad x = -2$$

where

$$x = 8 \text{ and } x = -2 \text{ are the roots of } x^2 - 6x - 16 = 0$$

Checking:

The original equation should always be used to verify a solution.

$$x^2 - 6x - 16 = 0$$

For: $x = 8$, $(8)^2 - 6(8) - 16 = 64 - 48 - 16 = 0$
For: $x = -2$, $(-2)^2 - 6(-2) - 16 = 4 + 12 - 16 = 0$
The roots satisfy the equation.

EXAMPLE 13-E:

Solve $2x^2 - 5x + 1 = 0$, using the method of completing the square.

Solution:

The method of completing the square requires, initially, that the coefficient of the second-degree term be equal to 1. This can be accomplished by dividing the original equation by the coefficient of the second-degree term if it is other than 1. Thus,

$$2x^2 - 5x + 1 = 0$$

becomes

$$\frac{2x^2}{2} - \frac{5x}{2} + \frac{1}{2} = \frac{0}{2}$$

or

$$x^2 - \frac{5x}{2} + \frac{1}{2} = 0$$

The procedure from this point on is identical to that used in Ex. 13-D. Transpose the constant term to the right.

$$x^2 - \frac{5x}{2} = -\frac{1}{2}$$

Divide $(-\frac{5}{2})$ by 2, square, and add to both sides.

$$\left(\frac{-\frac{5}{2}}{2}\right)^2 = \left(-\frac{5}{4}\right)^2 = \frac{25}{16}$$

Thus,

$$x^2 - \frac{5x}{2} + \frac{25}{16} = -\frac{1}{2} + \frac{25}{16} = \frac{17}{16}$$

Furthermore,

$$\left(x - \frac{5}{4}\right)^2 = \frac{17}{16}$$

Extracting square roots and solving for x again leads to completion

$$x - \frac{5}{4} = \pm\sqrt{\frac{17}{16}} = \pm\frac{\sqrt{17}}{4}$$

Hence,

$$x - \frac{5}{4} = \frac{\sqrt{17}}{4} \text{ and } x = \frac{5 + \sqrt{17}}{4}$$

$$x - \frac{5}{4} = -\frac{\sqrt{17}}{4} \text{ and } x = \frac{5 - \sqrt{17}}{4}$$

This is usually considered the acceptable form of the answer in the field of mathematics Engineers, however, may want the decimal equivalent Taking,

$$\sqrt{17} = 4\,123, \; x = \frac{5 + 4\,123}{4} = \frac{9\,123}{4} = 2\,281,$$

and

$$x = \frac{5 - 4\,123}{4} = \frac{0\,877}{4} = 0\,219$$

EXAMPLE 13 F

Find the roots of $x^2 - 10x + 29 = 0$

Solution

A brief study of that portion of the equation containing the variables $x^2 - 10x$ suggests that if $+25$ were to be added to these two terms, the resulting expression, $x^2 - 10x + 25$, would be a perfect trinomial square, where

$$x^2 - 10x + 25 = (x - 5)^2$$

Notice that the original quadratic can be written as

$$(x^2 - 10x + 25) + 4 = 0 \quad (25 + 4 = 29)$$

Thus, the perfect square trinomial is actually contained in the original equation It was just a matter of being able to recognize it and then to isolate it The designed intent of this example was to highlight this technique

The problem can now be completed by the procedures adopted previously

$$x^2 - 10x + 25 = -4$$
$$(x - 5)^2 = -4$$

Extracting square roots and solving for x leads to:

$$x - 5 = \pm\sqrt{-4} = \pm j2$$

where

$$x = 5 + j2$$

and

$$x = 5 - j2$$

These are called *imaginary roots*; hence, the solution of $x^2 - 10x + 29 = 0$ apparently contains no real root.

Perhaps a graph of the function may tend to clarify this condition (Fig. 13-3).

$$y = f(x) = x^2 - 10x + 29 = 0$$
$$f(0) = (0)^2 - 10(0) + 29 = 0 - 0 + 29 = 29$$
$$f(2) = (2)^2 - 10(2) + 29 = 4 - 20 + 29 = 13$$
$$f(-2) = (-2)^2 - 10(-2) + 29 = 4 + 20 + 29 = 53$$
$$f(3) = (3)^2 - 10(3) + 29 = 9 - 30 + 29 = 8$$
$$f(4) = (4)^2 - 10(4) + 29 = 16 - 40 + 29 = 5$$
$$f(5) = (5)^2 - 10(5) + 29 = 25 - 50 + 29 = 4$$
$$f(6) = (6)^2 - 10(6) + 29 = 36 - 60 + 29 = 5$$
$$f(8) = (8)^2 - 10(8) + 29 = 64 - 80 + 29 = 13$$
$$f(-1) = (-1)^2 - 10(-1) + 29 = 1 + 10 + 29 = 40$$

x	$y = f(x)$
0	29
2	13
−2	53
3	8
4	5
5	4
6	5
8	13
−1	40

It appears that the general form of the curve can be approximated from the tabulated coordinates.

From the graph it is evident that the curve does not cross the x-axis; hence, the function has no real roots. This is a geometric representation of a function without real roots; it is not, however, a graphical representation of imaginary numbers.

Another unique characteristic involving quadratics is the condition of a double root.

A perfect square trinomial of the form $x^2 - 10x + 25 = 0$, or $(x - 5)^2 = 0$, represents a function with a double root or two equal roots.

$$(x - 5)(x - 5) = 0$$

where

$$x - 5 = 0 \text{ and } x = 5$$
$$x - 5 = 0 \text{ and } x = 5$$

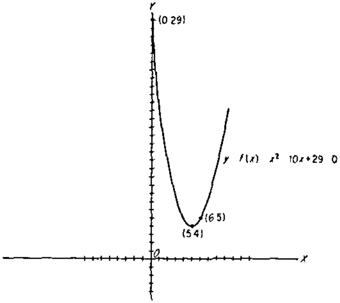


Figure 13-3

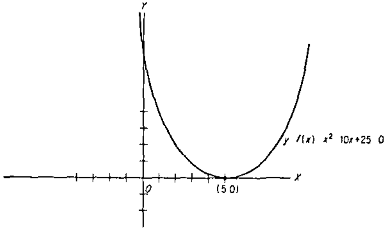


Figure 13-4

Geometrically, this indicates that the curve touches, or is tangent to, the axis. In this example (Fig. 13-4), the parabola touches the x-axis at (5,0).

EXERCISES 13-2

Solve the following quadratics by the method of completing the square. Where applicable, leave the answer in simplest radical form.

1. $x^2 - 4x - 21 = 0$	*2.* $y^2 + 12y = 64$
3. $x^2 + 2x - 35 = 0$	*4.* $y^2 = 18 - 3y$
5. $2x^2 - 12x + 18 = 0$	*6.* $4x^2 - 16x + 7 = 0$
7. $3x^2 - 5x + 1 = 0$	*8.* $2x^2 - 3x = 3$
9. $5x^2 - 4x = 5$	*10.* $5x^2 - 4x + 5 = 0$
11. $6y^2 + 18y - 7 = 0$	*12.* $25y^2 + 30y + 8 = 0$

13. $9x^2 + 12x + 2 = 0$

14. $2t^2 - 5t = 4$

15. $4x^2 - 2x = \dfrac{3}{4}$

13-3 QUADRATIC FORMULA

If the principle of completing the square is applied to the standard form of the quadratic, an equation will be developed which is called the **quadratic formula**. The quadratic formula is considered one of the most important relationships in algebra. It provides a method of obtaining roots by substituting numbers (in the formula) affiliated with the terms of a quadratic equation.

As indicated, the formula will be derived by completing the square of $ax^2 + bx + c = 0$

$$ax^2 + bx = -c$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$$

Thus,

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

and

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which is called the *quadratic formula*, where

a is the coefficient of the second-degree term,
b is the coefficient of the linear term, and
c is the constant term.

EXAMPLE 13-G:

Solve $6x^2 - 11x = 35$, using the quadratic formula.

Solution:

Arrange terms to correspond to standard form:

$$6x^2 - 11x - 35 = 0$$

Thus,

$$a = 6,\ b = -11, \text{ and } c = -35$$

The values of a, b, and c are substituted, respectively, in the quadratic formula The resulting expression is simplified and the roots are determined accordingly

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(6)(-35)}}{2(6)} = \frac{11 \pm \sqrt{121 + 840}}{12}$$

$$= \frac{11 \pm \sqrt{961}}{12} = \frac{11 \pm 31}{12}$$

Thus the roots are

$$x = \frac{11 + 31}{12} = \frac{42}{12} = \frac{7}{2}$$

and

$$x = \frac{11 - 31}{12} = -\frac{20}{12} = -\frac{5}{3}$$

Check by substitution:

$$x = \frac{7}{2};\ 6x^2 - 11x - 35 = 6\left(\frac{7}{2}\right)^2 - 11\left(\frac{7}{2}\right) - 35$$

$$= \frac{147}{2} - \frac{77}{2} - 35 = \frac{70}{2} - 35 = 0$$

$$x = -\frac{5}{3},\ 6\left(-\frac{5}{3}\right)^2 - 11\left(-\frac{5}{3}\right) - 35 = \frac{50}{3} + \frac{55}{3} - 35 = \frac{105}{3} - 35 = 0$$

Hence, the equation is satisfied and the roots have been determined.

EXAMPLE 13-H:

Using the quadratic formula, find the roots of $3x^2 + 5x + 4 = 0$.

Solution:

$a = 3$, $b = 5$, and $c = 4$ Substituting accordingly into the quadratic formula leads to.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(5) \pm \sqrt{(5)^2 - 4(3)(4)}}{2(3)} = \frac{-5 \pm \sqrt{25 - 48}}{6} = \frac{-5 \pm \sqrt{-23}}{6}$$

from which,

$$x = \frac{-5 + j\sqrt{23}}{6}$$

and

$$x = \frac{-5 - j\sqrt{23}}{6}$$

These of course are imaginary roots. One of the roots will be checked: $x = \dfrac{-5 - j\sqrt{23}}{6}$.

$$3x^2 + 5x + 4 = 3\left(\frac{-5 - j\sqrt{23}}{6}\right)^2 + 5\left(\frac{-5 - j\sqrt{23}}{6}\right) + 4$$

$$= 3\left(\frac{25 + 10j\sqrt{23} + j^2 23}{36}\right) - \frac{25}{6} - \frac{5j\sqrt{23}}{6} + 4$$

$$= \frac{25}{12} + \frac{5j\sqrt{23}}{6} - \frac{23}{12} - \frac{25}{6} - \frac{5j\sqrt{23}}{6} + 4$$

$$= \frac{25}{12} - \frac{23}{12} - \frac{50}{12} + \frac{48}{12} = \frac{25 + 48 - 23 - 50}{12} = \frac{73 - 73}{12} = 0$$

Thus, the equation is satisfied.

EXERCISES 13-3

Solve by quadratic formula. Leave the answers in simplified radical form.

1. $x^2 - 3x - 4 = 0$

2. $y^2 - 5y + 4 = 0$

3. $5x - 6 + 6x^2 = 0$

4. $12 - 7y - 10y^2 = 0$

5. $12x^2 - 9x = 30$

6. $Ax^2 + Bx + C = 0$

7. $x^2 - 2x + 2 = 0$

8. $\dfrac{1}{x+1} + \dfrac{2}{x-1} = 3$

9. $7x^2 + 4x - 20 = 0$

10. $4x^2 = 48 + 16x$

11. $8x^2 + 6x - 37 = 2x^2 + 35$

12. $7x^2 - 11x + 4 = 0$

13. $2x^2 - 5x - 4 = 0$

14. $\dfrac{2x^2}{9} + \dfrac{x}{15} - \dfrac{4}{25} = 0$

15. $\dfrac{x^2}{9} - \dfrac{4x}{15} + \dfrac{4}{25} = 0$

13-4 EQUATIONS WITH RADICALS

Equations may very often appear with terms that contain radicals. Before this type of an equation can be solved, the radicals have to be removed. This is usually done by re-arranging terms and squaring both sides. Sometimes the process has to be repeated.

This procedure will eventually transform the given equation into a manageable linear equation or quadratic equation. At the same time, this

technique may introduce extraneous roots Extraneous roots are additional roots that will not satisfy the original equation Thus, it becomes extremely important to check all roots before assuming that they are part of the solution

EXAMPLE 13-1

Solve the equation $\sqrt{3x+1} - \sqrt{2x-1} = 1$

Solution

Transpose the second radical to the right side of the equation and square both sides

$$\sqrt{3x+1} - \sqrt{2x-1} = 1$$
$$\sqrt{3x+1} = 1 + \sqrt{2x-1}$$

Squaring

$$3x + 1 = 1 + 2\sqrt{2x-1} + (2x - 1)$$

Collecting like terms

$$3x + 1 - 1 - 2x + 1 = 2\sqrt{2x-1}$$
$$x + 1 = 2\sqrt{2x-1}$$

Squaring again

$$x^2 + 2x + 1 = 4(2x - 1)$$

from which,

$$x^2 - 6x + 5 = 0$$

Factoring

$$(x - 5)(x - 1) = 0$$

leads to

$$x - 5 = 0 \text{ and } x = 5$$
$$x - 1 = 0 \text{ and } x = 1$$

Checking in the original equation

$$x = 5, \sqrt{3x+1} - \sqrt{2x-1} = 1$$
$$\sqrt{3(5)+1} - \sqrt{2(5)-1} = \sqrt{16} - \sqrt{9} = 4 - 3 = 1$$

The equation is satisfied, thus, $x = 5$ is a root

$$x = 1, \sqrt{3(1)+1} - \sqrt{2(1)-1} = \sqrt{4} - \sqrt{1} = 2 - 1 = 1$$

and again the equation is balanced, indicating that both $x = 5$ and $x = 1$ are roots

EXAMPLE 13-J:

Solve $\sqrt{5x+6} = 3 + \sqrt{x+3}$

Solution:

Square both sides.

$$5x + 6 = 9 + 6\sqrt{x+3} + x + 3$$

Collect like terms,

$$4x - 6 = 6\sqrt{x+3}$$

Divide by 2,

$$2x - 3 = 3\sqrt{x+3}$$

Squaring again,

$$4x^2 - 12x + 9 = 9(x+3) = 9x + 27$$

Combine like terms,

$$4x^2 - 21x - 18 = 0$$

Solve the equation by way of the quadratic formula.

$$x = \frac{21 \pm \sqrt{(21)^2 - 4(4)(-18)}}{8}$$

$$= \frac{21 \pm \sqrt{441 + 288}}{8} = \frac{21 \pm \sqrt{729}}{8}$$

$$= \frac{21 \pm 27}{8}$$

from which

$$x = \frac{21 + 27}{8} = \frac{48}{8} = 6$$

and

$$x = \frac{21 - 27}{8} = -\frac{6}{8} = -\frac{3}{4}$$

Checking:

$$x = 6;\ \sqrt{5x+6} = 3 + \sqrt{x+3}$$

$$\sqrt{5(6)+6} = 3 + \sqrt{6+3}$$

$$\sqrt{36} = 3 + \sqrt{9} \text{ and } 6 = 3 + 3$$

Thus, $x = 6$ is a root.

$$x = -\frac{3}{4};\ \sqrt{5\left(-\frac{3}{4}\right) + 6} = 3 + \sqrt{-\frac{3}{4} + 3}$$

$$\sqrt{\frac{-15 + 24}{4}} = 3 + \sqrt{\frac{-3 + 12}{4}}$$

where, $\sqrt{\frac{9}{4}} \neq 3 + \sqrt{\frac{9}{4}}$

and the equation is not satisfied Hence, $x = -\frac{3}{4}$ is an extraneous root

Extraneous roots may be introduced whenever an equation is multiplied by a term containing the variable

EXAMPLE 13-K

Solve $5 - \frac{x^2}{x-2} + \frac{4}{x-2} + 1 = 0$

Solution

Multiply through by $(x - 2)$

$$5(x-2) - (x-2)\frac{x^2}{x-2} + (x-2)\frac{4}{x-2} + 1(x-2) = 0(x-2)$$

Removing parantheses

$$5x - 10 - x^2 + 4 + x - 2 = 0$$

Combining like terms

$$-x^2 + 6x - 8 = 0, \text{ or } x^2 - 6x + 8 = 0$$

Factoring

$$(x-4)(x-2) = 0$$

thus,

$$x = 4 \text{ and } x = 2$$

Checking

$$x = 2, 5 - \frac{x^2}{x-2} + \frac{4}{x-2} + 1 = 0$$

$$5 - \frac{(2)^2}{2-2} + \frac{4}{2-2} + 1 = 0$$

$$5 - \frac{4}{0} + \frac{4}{0} + 1 \neq 0$$

This statement cannot be evaluated, because division by zero is undefined, thus, $x = 2$ must be considered an extraneous root

$$x = 4, 5 - \frac{(4)^2}{4-2} + \frac{4}{4-2} + 1 = 0$$

$$5 - \frac{16}{2} + \frac{4}{2} + 1 = 5 - 8 + 2 + 1 = 0$$

and the equation is satisfied, hence, $x = 4$ is the only root

Although the process of multiplying an equation by a term or factor containing the variable may introduce an extraneous root, it may also suggest that the original equation was not in simplified form to begin with. On the other hand dividing by a term or factor containing the variable may result in a root being lost. This root is defined as a *vanishing root*. In any event, *checking roots* must be considered an integral step in the procedure for solving an equation.

EXAMPLE 13-L:

Solve $x(x-2)(x-3) = 4(x-2)$

Solution:

Divide through by $(x-2)$

$$\frac{x(x-2)(x-3)}{(x-2)} = \frac{4(x-2)}{(x-2)}$$

leading to

$$x(x-3) = 4$$

Combining terms:

$$x^2 - 3x - 4 = 0$$

Factoring:

$$(x-4)(x+1) = 0$$

where

$$x = 4 \text{ and } x = -1$$

Checking:

$$\begin{aligned} x = 4; \qquad & x(x-2)(x-3) = 4(x-2) \\ & 4(4-2)(4-3) = 4(4-2) \\ & 4(2)(1) = 4(2), \text{ satisfied.} \\ x = -1 \qquad & -1(-1-2)(-1-3) = 4(-1-2) \\ & -1(-3)(-4) = 4(-3) \end{aligned}$$

Again the equation is satisfied. There is another root that will satisfy the original equation, $x = 2$.

$$2(2-2)(2-3) = 4(2-2)$$
$$0 = 0$$

The root that was lost (or vanished) as a result of dividing the original equation by $(x-2)$ is $x = 2$. Notice that $x(x-2)(x-3) = 4(x-2)$ and $x(x-3) = 4$ are not equivalent equations. The first equation is a cubic (3 roots) whereas the second is a quadratic (2 roots).

Solve the given equations and check for extraneous roots

1. $\sqrt{x-5}=3$

2. $6-\sqrt{2x+3}=0$

3. $\sqrt{x^2-4}-\sqrt{12}=0$

4. $\sqrt{x+2}=\sqrt{x^2}$

5 $3\sqrt{3x-2}+\sqrt{3x-2}=20$

6. $\sqrt{x+6}-\sqrt{3x-5}=1$

7. $\sqrt{x^2-8}=2\sqrt{x^2-14}$

8. $\sqrt{5x+7}-2\sqrt{x-1}=\sqrt{3}$

9. $\sqrt{8x+9}-\sqrt{4x+5}=2$

10. $\sqrt{6x-5}-\sqrt{3x-2}=\sqrt{x-5}$

11. $(x-3)(x+3)=2(x+3)(x-5)$

12. $\dfrac{1}{x-3}+\dfrac{x}{x+3}=\dfrac{6}{x^2-9}$

13. $\sqrt{x^2+14x+50}=1$

14. $(x^2+6x+10)-2\sqrt{x^2+6x+10}+1=0$

15. $\sqrt{8x^2-41x+32}=3(2-x)$

13-5 SYSTEMS OF QUADRATIC EQUATIONS

A system of quadratic equations in two variables, can be solved by using the procedures adopted for systems of linear equations Each system will have to be studied and then solved by the method that seems to be most appropriate A quadratic equation in two variables usually represents a parabola, circle, ellipse, or hyperbola

EXAMPLE 13 M

Solve the system of equations and check graphically

$$9x^2+25y^2=225$$
$$x^2+y^2=9$$

Solution

Multiply the second equation by -9 and add it to the first

$$\begin{array}{r} 9x^2+25y^2=225 \\ -9x^2-9y^2=-81 \\ \hline 16y^2=144 \end{array}$$

Solving for y

$$y=\pm\sqrt{\frac{144}{16}}=\pm 3$$

Substitute the values of y in either equation and solve for the corresponding values of x

$$x^2 + y^2 = 9$$

$$y = 3, \quad x^2 + (3)^2 = 9, \quad x^2 + 9 = 9, \text{ and } x^2 = 0, \text{ or } x = 0$$

$$y = -3, x^2 + (-3)^2 = 9, x^2 + 9 = 9, \text{ and } x = 0$$

Thus, the solution of the system is (0, 3) and (0, −3). The graphical or geometric solution appears in Fig. 13-5.

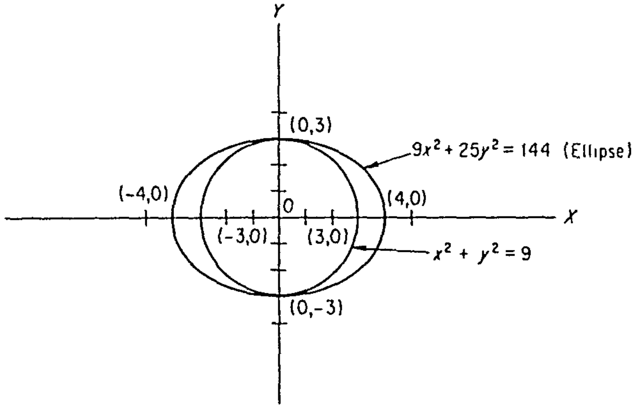


Figure 13-5

EXAMPLE 13-N:

Solve the system of equations and represent graphically.

$$16x^2 - 9y^2 = 144$$

$$y^2 = \frac{128}{9}x$$

Solution:

Substitute $y^2 = (128/9)x$ into the first equation and solve for x.

$$16x^2 - 9\left(\frac{128x}{9}\right) = 144$$

$$16x^2 - 128x - 144 = 0$$

Divide through by 16.

$$\frac{16x^2}{16} - \frac{128x}{16} - \frac{144}{16} = \frac{0}{16}$$

or

$$x^2 - 8x - 9 = 0$$

Factor:

$$(x - 9)(x + 1) = 0$$

from which

$$x - 9 = 0 \text{ and } x = 9$$

$$(x + 1) = 0 \text{ and } x = -1$$

Solve for the corresponding values of y

$$x = 9, y^2 = \frac{128}{9}x = \frac{128}{9}(9) = 128$$

and

$$y = \pm\sqrt{128} = \pm\sqrt{64 \cdot 2} = \pm 8\sqrt{2}$$

for

$$x = -1, y^2 = \frac{128}{9}(-1) = -\frac{128}{9}$$

which means that there are no real roots when $x = -1$ The graph will clarify this condition (Fig 13-6)

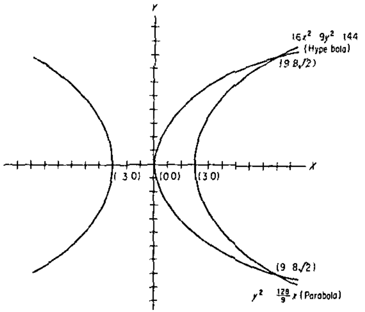


Figure 13-6

Thus, the solution of this system is $(9, 8\sqrt{2})$ and $9, -8\sqrt{2})$

Graphically, the solution appears as the intersection of a parabola and a hyperbola (Fig 13 6)

Notice that the function $y^2 = (128/9)x$ exists only when x is positive ($x > 0$) Thus there can be no solution for $x = -1$, because the parabola is limited to the first and fourth quadrants (no intersections possible in second and third quadrants)

EXERCISES 13-5

Solve the systems of equations and verify by sketch

1. $x = y$
$x^2 = 4y$

2. $xy = 1$
$x = y$

3. $y^2 - 2x = 0$
$xy + 4 = 0$

4. $x^2 + y^2 = 16$
$x - y + 4 = 0$

5. $4y^2 + 9x^2 = 36$
$3x - 2y = 6$

6. $16y^2 + 25x^2 = 400$
$x^2 + y^2 = 16$

7. $16y^2 + 25x^2 = 400$
$9x^2 + 16y^2 = 144$

8. $9x^2 + 16y^2 = 144$
$9x^2 - 16y^2 = 144$

9. $16y^2 + 36x^2 = 576$
$2y^2 + 27x = 0$

10. $y^2 = 16x$
$y^2 + 16x = 64$

11. $y^2 = 16x$
$x^2 = 16y$

12. $x^2 + y^2 = 64$
$x^2 = -12y$

REVIEW EXERCISES 13-6

The problems in this section deal with engineering-scientific formulas involving some of the mathematical concepts just covered.

Given a formula and various quantities associated with the formula, find the value of one of its elements.

1. $t = 2\pi\sqrt{l/g}$; t is the period, in seconds, of a pendulum of length l in meters, m.

$$g = 9.8\,\frac{m}{\text{sec}^2},\ \pi = 3.14$$

Find l if $t = 2$ sec.

2. $v = \sqrt{2gs}$; v is the final velocity in feet per second of a body that has fallen s ft.

$$g = 32.2\ \text{ft/sec}^2.$$

Find the distance, s, that a body must fall to reach a velocity of 128.8 ft/sec.

3. $m = m_t/\sqrt{1 - v^2/c^2}$; mass, m, varies with velocity, v; m_t is the mass of a body at rest ($v = 0$); and c is the speed of light ($c = 3 \times 10^8$ m/sec). If $m_t = 100$ g, at what velocity will the mass double ($m = 200$g)?

4. $Z = \sqrt{R^2 + (X_L - X_C)^2}$; Z is the impedance of a circuit with a resistance, R, an inductive reactance, X_L, and a capacitive reactance, X_C.
Find X_C if $Z = 25$ ohms, $R = 24$ ohms, and $X_L = 9$ ohms.

5. Find X_L if $Z = 50$ ohms, $R = 40$ ohms, and $X_C = 5$ ohms.

6. $T = (wl/2)\sqrt{1 + l^2/16d^2}$ gives the tension in a suspended support cable, where w is the load per foot, l is the span of the cable, and d is the sag at the mid-point.
Find T if $w = 1{,}000$ lb/ft, $l = 640$ ft, and $d = 40$ ft.

7. Solve for d in the formula of exercise 6.

8. Find the sag, d, if $T = 1{,}280{,}000$ lb, $w = 800$ lb/ft, and $l = 800$ ft.

9. $d = \sqrt{h^2 + (b_1 + b_2)^2/4}$ is the formula for the diagonal of an isosceles trapezoid (Fig. 13-7).
Find b_2 if $d = 13$ in., $h = 5$ in., and $b_1 = 14$ in.

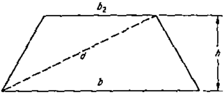


Figure 13-7

10. $l = 2\sqrt{2rh - h^2}$, where l is the length of a chord, r is the radius, and h is the distance of the chord from the circle (Fig 13-8)
If $r = 10$ in and $l = 8$ in, find h

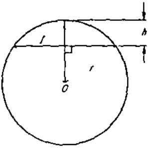


Figure 13-8

11. $V = \pi h(D^2 - d^2)/4$ is the formula for the cross section volume of a pipe with an outside diameter of D, an inside diameter of d, and a height (or length) of h
Find d if $V = 180$ in ^{3}, $D = 3$ in, and $h = 144$ in

12. $d = \sqrt{2(\sqrt{2} + 1)A}/2$ is the formula for a diagonal of an octagon with area A Find A if $d = 8$ (Fig 13-9)

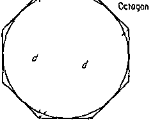


Figure 13-9

UNIT III

Advanced Topics

Perhaps the oldest branch of mathematics can be associated with geometry. Geometry is a subject most essential to those who are pursuing areas of specialization involving the elements of surveying and the design of machine and structural members.

This unit includes selected topics from plane and solid geometry along with an introduction to analytic geometry. Included also is the treatment of trigonometry, which further leads to complex numbers. Complex numbers are involved with the study of a-c circuits and are an important consideration for the electronic technician.

CHAPTER 14

Logarithms

Logarithms can be classified as one of the mathematical innovations that expedite tedious arithmetic operations. Recently, high-speed electrical calculators and electronic computors have minimized the use of logarithms. There are several areas, however, in which this device is still employed. Many equations involving the behavior of steam and other gases carry numerical exponents that are not integers; $H = k(t_s - t_1)^{1.6}$, for example. With the use of logarithms, this type of computation is reduced to a simple operation in multiplication.

Another sector in which the technician might rely on logarithms is in the area of extracting roots other than square roots.

14-1 DEFINITIONS

The logarithm (log) of a number is the exponent of the power to which a second number, called the base, must be raised to equal the given number.

$$\mathbf{10^2 = 100 \text{ or } \log_{10} 100 = 2}$$

which is stated as the logarithm of 100, to the base 10, is equal to 2. The base must be positive and a number other than 1.

Primarily, two systems of logarithms have been developed: **common logarithms**, with a **base 10**, and **natural logarithms**, with a **base e**, where $e = 2.718$. Whenever the base is not indicated, it is understood to be 10. In general form,

$$\log_b N = a, \text{ where } N = b^a$$

The first statement is referred to as **logarithmic form**, whereas the second is called the **exponential form** ($b > 0$ and $b \neq 1$).

Exponential Form	*Logarithmic Form*
$10^1 = 10$	$\log_{10} 10 = 1$
$5^3 = 125$	$\log_5 125 = 3$
$7^4 = 2{,}401$	$\log_7 2{,}401 = 4$
$4^{1/2} = 2$	$\log_4 2 = \frac{1}{2}$
$16^{3/2} = 64$	$\log_{16} 64 = \frac{3}{2}$
$15^{-1} = \frac{1}{15}$	$\log_{15} \frac{1}{15} = -1$
$8^{-2/3} = \frac{1}{4}$	$\log_8 \frac{1}{4} = -\frac{2}{3}$
$25^{-2} = \frac{1}{625}$	$\log_{25} \frac{1}{625} = -2$
$12^0 = 1$	$\log_{12} 1 = 0$
$e^0 = 1$	$\log_e 1 = 0$
$10^0 = 1$	$\log_{10} 1 = 0$

The last three illustrations demonstrate that the logarithm of 1 to any base is equal to zero

$$\log_b 1 = 0 \text{ or } b^0 = 1$$

Figure 14-1 is a sketch of the logarithmic curve $y = \log_b x$

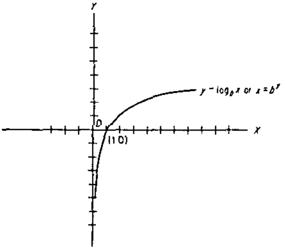


Figure 14-1

EXERCISES 14-1

Express in exponential form (Ex 1-10)

1. $\log_{10} 10{,}000 = 4$

2. $\log_{100} 10{,}000 = 2$

3. $\log_{13} 2{,}197 = 3$
4. $\log_2 64 = 6$
5. $\log_8 1 = 0$
6. $\log_{10} \frac{1}{100} = -2$
7. $\log_{27} 3 = \frac{1}{3}$
8. $\log_{32} 8 = \frac{3}{5}$
9. $\log_{343} \frac{1}{49} = -\frac{2}{3}$
10. $\log_e N = X$

Express in logarithmic form (Ex. 11-20).

11. $4^2 = 16$
12. $10^5 = 100{,}000$
13. $10^{-1} = \frac{1}{10}$
14. $19^0 = 1$
15. $25^{3/2} = 125$
16. $125^{2/3} = 25$
17. $64^{-1/2} = \frac{1}{8}$
18. $64^{-2} = \frac{1}{4{,}096}$
19. $e^a = N$
20. $3b^{2a} = M$

Find the value of each of the following logarithms (Ex. 21-28).

21. $\log_5 125$
22. $\log_7 49$
23. $\log_7 343$
24. $\log_{64} 8$
25. $\log_8 64$
26. $\log_2 2^2$
27. $\log_3 1$
28. $\log_{12} \frac{1}{12}$

Find the value of the base, b (Ex. 29-36).

29. $\log_b 100 = 2$
30. $\log_b 81 = 2$
31. $\log_b 27 \times 10^3 = 3$
32. $\log_b \frac{1}{529} = -2$
33. $\log_b 6 = \frac{1}{2}$
34. $\log_b 6 = -\frac{1}{2}$
35. $\log_b 8 = \frac{3}{2}$
36. $\log_b 1 = 0$

14-2 PROPERTIES OF LOGARITHMS

Logarithms, by definition, are exponents and in terms of mathematical operations, conform to the laws of exponents.

Laws of Exponents	*Properties of Logarithms*
$a^m \cdot a^n = a^{m+n}$	$\log_b MN = \log_b M + \log_b N$
$\frac{a^m}{a^n} = a^{m-n}$	$\log_b \frac{M}{N} = \log_b M - \log_b N$

$$(a^m)^n = a^{mn} \qquad \log_b M^n = n \log_b M$$

$$\log_b \sqrt[n]{M} = \log_b M^{1/n} = \frac{1}{n} \log_b M$$

Stated as rules

$$\log_b MN = \log_b M + \log_b N$$

The logarithm of a product is equal to the sum of the logarithms of the separate factors

$$\log_b (\pi)(4\ 750) = \log_b \pi + \log_b 4\ 750$$

The logarithm of a quotient is equal to the difference of the logarithm of the dividend and the logarithm of the divisor

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b \frac{\pi}{4\ 750} = \log_b \pi - \log_b 4\ 750$$

It should be pointed out that the sum of two logarithms is not the same as the logarithm of a sum

$$\log_b (M + N) \neq \log_b M + \log_b N$$

Likewise,

$$\log_b (M - N) \neq \log_b M - \log_b N$$

The logarithm of a power of a number is equal to the product of the exponent and the logarithm of the number

$$\log_b M^n = n \log_b M$$

$$\log_b \sqrt[n]{M} = \frac{1}{n} \log_b M$$

$$\log_b \sqrt[3]{4\ 750} = \frac{1}{3} \log_b 4\ 750$$

The properties of logarithms provide a computational technique that is less trying than the rigors of arithmetic A product of the order (3 14)(77,321 73)(36 35) reduces to the sum of the logarithms and of the separate factors $\log_b$ (3 14)(77,321 73)(36 35) $= \log_b 3\ 14 + \log_b 77{,}321\ 73 + \log_b 36\ 35$ Obviously, with an office calculator the original problem could be completed within seconds On the other hand a problem such as $(27\ 93)^{1\ 17}$ would require equipment that may not be readily available in some engineering offices Here, the use of logarithms provides an accurate and routine solution

$$(27\ 93)^{1\ 17} = N$$

If two numbers are equal their logarithms are equal; thus:

$$\log_b N = \log_b (27.93)^{1.17} = 1.17 \log_b 27.93$$

Before this computation can be completed, the method for determining the logarithm of a number must be established.

14-3 CHARACTERISTIC AND MANTISSA

The logarithm of a number is made up of two parts, the *characteristic* and *mantissa*. *The characteristic is an integer, either positive, negative, or zero, and is determined by inspection, whereas the mantissa is, by and large, an unending decimal, determined through use of a table.* A further study of the definition of a logarithm will perhaps lead to an understanding of the characteristic and mantissa.

Discussion involving logarithms, from this point on, will be directed toward logarithms of base 10, unless noted otherwise. Thus, $\log_{10} N$ will be written as $\log N$.

Exponential form	Logarithmic form
$10^5 = 100{,}000$	$\log 100{,}000 = \log 10^5 = 5$
$10^4 = 10{,}000$	$\log 10{,}000 = \log 10^4 = 4$
$10^3 = 1{,}000$	$\log 1{,}000 = \log 10^3 = 3$
$10^2 = 100$	$\log 100 = \log 10^2 = 2$
$10^1 = 10$	$\log 10 = \log 10^1 = 1$
$10^0 = 1$	$\log 1 = \log 10^0 = 0$
$10^{-1} = \frac{1}{10} = 0.1$	$\log 0.1 = \log 10^{-1} = -1$
$10^{-2} = \frac{1}{100} = 0.01$	$\log 0.01 = \log 10^{-2} = -2$
$10^{-3} = \frac{1}{1{,}000} = 0.001$	$\log 0.001 = \log 10^{-3} = -3$
$10^{-4} = \frac{1}{10{,}000} = 0.0001$	$\log 0.0001 = \log 10^{-4} = -4$
$10^{-5} = \frac{1}{100{,}000} = 0.00001$	$\log 0.00001 = \log 10^{-5} = -5$

Figure 14-2

First of all, it should be pointed out (and emphasized) that multiples of 10 have logarithms that are integers. This would further suggest that additional information might be needed to determine the logarithms of numbers that are not multiples or powers of 10. This information has been compiled and appears as Table II in the Appendix. **Table II** is **called** a **four-place table of logarithms.** The data contained in this table is the decimal part (mantissa) of the logarithm and has been computed and rounded off to four significant figures. Other tables are available to five or six decimal places.

If logarithms are to be carried to four decimal places, then it follows that log 10,000 should be written as. 4.0000, log 1,000 = 3.0000, and log 10 = 1.0000, and so on. Referring to log 10,000 = 4.0000, **4** is called the **characteristic** and **.0000** the **mantissa.** The characteristic can be either positive or negative, as well as zero; however, the mantissa is always considered

positive Notice also that the mantissa is the same for 10,000, for 1,000, as well as for 10 This leads to an important concept, which is summarized within the following statement *The mantissa of a logarithm depends on the sequence of the digits in the number and is not affected by the position of the decimal point* Hence, log 3 14, log 0 0314, and log 3140 0 will all have the same mantissa, the characteristic, on the other hand, will change accordingly

14-4 LOGARITHM OF A NUMBER

It was stated earlier that the characteristic of a logarithm could be determined by inspection This method will be demonstrated along with the procedure of finding the logarithm of a number Three numbers, 16,200, 16 2 and 0 0162, will be used in that order

EXAMPLE 14 A

Find the log 16,200

Solution

Referring to the table in Fig 14-2, it is apparent that 100,000 > 16,200 > 10,000, which indicates that the exponent of the power to which 10 must be raised, to equal 16,200, is greater than 4 but less than 5

$$10^a = 16{,}200, \text{ where } 4 < a < 5$$

This may be clarified somewhat if 16,200 is inserted in the table in Fig 14-2

Exponential Form	*Logarithmic Form*	*Characteristic*
$100{,}000 = 10^5$	$\log 10^5 = 5\ 0000$	5
$16{,}200 = 10^4(1\ 62)$	$\log 10^4(1\ 62) = \log 10^4 + \log 1\ 62$	4
	$= 4 + \log 1\ 62$	
$10{,}000 = 10^4$	$\log 10^4 = 4\ 000$	4

Since

$$\log 1 < \log 1\ 62 < \log 10$$

or

$$0 < \log 1\ 62 < 1$$

it follows that

$$\log 10^4 < \log 16{,}200 < \log 10^5$$

and

$$4\ 0000 < \log 16{,}200 < 5\ 0000$$

which states that the logarithm of 16,200, (log 16,200) is equal to 4 plus a decimal, where 4 is the characteristic The decimal portion, the mantissa, can be located in Table II, which is partially reproduced in Fig 14-3

For the log M, column N represents the first two digits of the number M, whereas the other columns, 0, 1, 2, 3, , represent the appropriate third digit of the number M If M contains more than three significant figures, it

N	0	1	2	3	4
15	1761	1790	1818	1847	...
16	2041	2068	**2095**	2122	...
17	2304	2330	2355	2380	...

Figure 14-3

will have to be rounded off to three digits, for the time being. Hence, for 1.62 or 16.2 or 162, . . . , the first two digits, 16, are located under N and the third is located under the column headed 2. Thus, the mantissa of 162 or 16.2 or 1.62, . . . , is .2095. Therefore, with characteristic and mantissa both defined, the logarithm of 16,200 can be written:

$$\log 16{,}200 = 4.2095 \text{ or } 10^{4.2095} \approx 16{,}200$$

EXAMPLE 14-B:

Find the log 16.2.

Solution:

Exponential Form	*Logarithmic Form*	*Characteristic*
$100 = 10^2$	$\log 10^2 = 2.0000$	2
$16.2 = 10^1(1.62)$	$\log 10^1(1.62) = \log 10 + \log 1.62$	
	$= 1 + \log 1.62$	1
$10 = 10^1$	$\log 10^1 = 1.0000$	1

This is analagous to the preceding illustration, Ex. 14-A.

Log $10 < \log 16.2 < \log 100$ and $1.0000 < \log 16.2 < 2.0000$, which states that log 16.2 is equal to 1 plus a decimal, where 1 is the characteristic. The mantissa (decimal portion) is found in Table II. Since the sequence of digits in the number 16.2 are the same as 1.62 or 162, the mantissa is equal to .2095, as previously determined (Ex. 14-A). Thus, $\log 16.2 = 1.2095$ or $10^{1.2095} = 16.2$.

EXAMPLE 14-C:

Find log 0.0162.

Solution:

The same procedure will be used here as in the preceding examples, 14-A and 14-B.

Exponential Form	*Logarithmic Form*	*Characteristic*
$0.100 = \frac{1}{10} = 10^{-1}$	$\log 10^{-1} = -1.0000$	-1
$0.0162 = \frac{1.62}{100} = 10^{-2}(1.62)$	$\log 10^{-2}(1.62) = \log 10^{-2} + \log 1.62$	
	$= -2 + \log 1.62$	-2
$0.010 = \frac{1}{100} = 10^{-2}$	$\log 10^{-2} = -2.0000$	-2

which would indicate that

$$\log 10^{-2} < \log 10^{-2}(1\ 62) < \log 10^{-1}$$

and

$$-2\ 0000 < \log 10^{-2}(1\ 62) < -1\ 0000$$

The mantissa is again 2095 and is considered positive The characteristic, on the other hand, is -2 Thus, log 0 0162 is equal to -2 plus 0 2095 Since the mantissa is positive, a notation of the form -2 2095 suggests that the entire quantity is negative Hence, when the characteristic is negative, an equivalent form is used whereby the minus sign does not appear in front of the logarithm of the number, as in the forms

$$\log 0\ 0162 = 8\ 2095 - 10 \qquad (8 - 10) = -2$$
$$\log 0\ 0162 = 18\ 2095 - 20 \qquad (18 - 20) = -2$$
$$\log 0\ 0162 = 28\ 2095 - 30 \qquad (28 - 30) = -2$$

rather than $\log 0\ 0162 = -2\ 2095$ (The last number 10, 20, 30, , is usually a multiple of 10)

The table in Fig 14 4 summarizes the results of the illustrative examples and lists several other logarithms as well

The characteristic of a logarithm can be determined by either of the following rules (using the table in Fig 14 4 as reference)

1 If the number N is written in scientific notation (powers of 10), the characteristic corresponds to the exponent of the power
2 For $N > 1$ the characteristic is positive and equal numerically to the number of digits to the left of the decimal point minus one

   For $N < 1$ the characteristic is negative and equal to the number of zeros preceding the first significant digit plus one

N	*Rule*	*Characteristic*
16,200	5(digits) $-$ 1 =	4
1020	4(digits) $-$ 1 =	3
0 00030	$-$[3(zeros) + 1] =	-4
0 821	$-$[0(zeros) + 1] =	-1

In the expression log 16,200 = 4 2095, 4 2095 is called the antilogarithm (antilog) of 16,200 The process of finding a number (antilog) corresponding to a given logarithm would be inverse of the procedure just established

EXAMPLE 14 D

(a) If $\log N = 2\ 1847$, find N

	N	Characteristic	Mantissa	Log N
	$16{,}200 = 1.62 \times 10^4$	4	.2095	4.2095
	$16.2 = 1.62 \times 10^1$	1	.2095	1.2095
	$0.0162 = 1.62 \times 10^{-2}$	-2	.2095	8.2095-10
	$1020 = 1.02 \times 10^3$	3	.0086	3.0086
	$5.0 = 5 \times 10^0$	0	.6990	0 6990
	$0.821 = 8.21 \times 10^{-1}$	-1	.9143	9.9143-10
Figure 14-4	$0.00030 = 3.0 \times 10^{-4}$	-4	.4771	6.4771-10

Solution:

The characteristic is 2, which means that $100 < N < 1{,}000$. From the abbreviated table (Fig. 14-3), the mantissa .1847 is found to correspond to 153.

Thus, if $\log N = 2.1847$, $N = 153$.

(b) Find the antilogarithm of $9.2304 - 10$.

Solution:

The characteristic is -1, which indicates that $0.1 < N < 1$. From the table, the mantissa .2304 is found to correspond to 170.

Thus, if $\log N = 9.2304 - 10$, $N = 0.170$.

EXERCISES 14-2

Find the logarithms of the following numbers (Ex. 1-15).

1. 6.

2. 60.

3. 0.60

4. 325.

5. 0.712

6. 899

7. 3.56×10^4

8. 3.14×10^{-1}

9. 0.00020

10. 12,300

11. $\frac{1}{4}$

12. $\sqrt{0.36}$

13. $(2.1)^2 \times 10^{-2}$

14. $\frac{1}{\sqrt{0.010}}$

15. $(10^{-3})(10^3)$

Find the antilogarithm of each of the logarithms (Ex. 15-25).

16. 0.0043

17. 1.0043

18. $9.0043 - 10$

19. 3.8129

20. 0.4771

21. $6.9309 - 10$

22. 2.0000

23. $13.0453 - 20$

24. 0.0000

25. 4.4440

If $\log 370 = 2\ 5682$, find the logarithms of the numbers listed (Ex 26-30)

26. 3 70 *27.* 37 0

28. 0 00370 *29.* 37×10^4

30. $\frac{37}{100}$

If the antilog $9\ 6031 - 10 = 0\ 401$, find the numbers corresponding to the logarithms listed (Ex 31-35)

31. 0 6031 *32.* 2 6031

33. $7\ 6031 - 10$ *34.* $12\ 6031 - 10$

35. $20\ 6031 - 20$

14-5 COMPUTATIONS

Modern techniques involving computations have to some extent minimized the importance of logarithms Logarithms, however, are very much a part of higher (theoretical) mathematics, especially natural logarithms Natural logarithms are also used in various branches of engineering to express thermal and other relationships Presently, the technician will appreciate the simplicity with which logarithms may facilitate the process of extracting roots and raising to a power The procedures leading to this operation involve, first, the fundamentals of arithmetic in application to logarithms Several examples will be used to demonstrate the procedure of adding and subtracting logarithms

EXAMPLE 14 E

If $\log M = 5\ 1676$ and $\log N = 3\ 3258$, find $\log M + \log N$

Solution

When the characteristics are positive, logarithms are added in the same manner as any other arithmetical numbers

$$\begin{aligned} \log M &= 5\ 1676 \\ \log N &= 3\ 3258 \\ \hline \log M + \log N &= 8\ 4934 \end{aligned}$$

EXAMPLE 14 F

If $\log M = 8\ 3562 - 10$ and $\log N = 3\ 9647$, find $\log M + \log N$

Solution:

When the characteristics are of opposite signs, the logarithms are combined algebraically Again, the mantissa remains positive

$$\begin{array}{r}\log M = 8.3562 - 10 \\ \log N = 3.9647 \\ \hline \log M + \log N = 12.3209 - 10 = 2.3209\end{array}$$

The same procedure (Ex. 14-F) also applies to the situation in which the characteristics are all negative.

Logarithms are combined in subtraction by following the procedures established for algebraic expressions. When the characteristic of the minuend is larger (absolute value) than the characteristic of the subtrahend the problem is routine. For some other conditions, however, it may be necessary, as a matter of convenience, to change the minuend to an equivalent logarithm whose characteristic is larger (absolute value) than the characteristic of the subtrahend. The concept of equivalent characteristics was demonstrated briefly in Ex. 14-C and will be re-emphasized again.

Log N	*Equivalent Forms of Log N*
1.2345	$2.2345 - 1$, $11.2345 - 10$, $21.2345 - 20, \ldots,$
0.2345	$10.2345 - 10$, $20.2345 - 20, \ldots,$
$8.3456 - 10$	$18.3456 - 20$, $48.3456 - 50$

EXAMPLE 14-G:

If $\log M = 3.2472$ and $\log N = 2.1395$, find $\log M - \log N$.

Solution:

Subtract 2.1395 from 3.2472, or

$$\begin{array}{r}\log M = 3.2472 \\ \log N = 2.1395 \\ \hline \log M - \log N = 1.1077\end{array}$$

EXAMPLE 14-H:

If $\log M = 8.5763 - 10$ and $\log N = 9.8732 - 10$, find $\log M - \log N$.

Solution:

Change $\log M$ to an equivalent logarithm such that its characteristic is larger (absolute value) than the subtrahend, $9.8732 - 10$.

$\log M = 8.5763 - 10 = 18.5763 - 20$. Next, subtract accordingly:

$$\begin{array}{r}\log M = 18.5763 - 20 \\ \log N = 9.8732 - 10 \\ \hline \log M - \log N = 8.7031 - 10\end{array}$$

Logarithms are added and subtracted, by and large, to carry out some

other operation involving logarithms, such as multiplication or division $\log MN = \log M + \log N$, or $\log M/N = \log M - \log N$

EXAMPLE 14 I

Find the value of $3\ 12 \times 921/0\ 098$, using logarithms

Solution

First, find the logarithms of the various numbers

$$\log 3\ 12 = 0\ 4942$$
$$\log 921 = 2\ 9643$$
$$\log 0\ 098 = 8\ 9912 - 10$$

Next, apply principles of logarithms and carry out arithmetic operations

$$\begin{aligned}\log \frac{3\ 12 \times 921}{0\ 098} &= \log 3\ 12 + \log 921 - \log 0\ 098 \\ &= 0\ 4942 + 2\ 9643 - (8\ 9912 - 10) \\ &= 3\ 4585 - (8\ 9912 - 10) \\ &= 4\ 4673\end{aligned}$$

The result, 4 4673, is in logarithmic form Completing the numerical solution of the given problem requires finding the antilog 4 4673 By inspec tion the characteristic is 4, and from Table II, the mantissa nearest to 4673 corresponds to $N = 293$ Thus, antilog 4 4673 = 29,300 Hence

$$\frac{3\ 12 \times 921}{0\ 098} = 29{,}300$$

EXAMPLE 14 J

Using the process of logarithms, find $\sqrt[3]{9{,}260}$

Solution

$\log \sqrt[n]{M} = 1/n \log \sqrt{M}$ is the principle that will lead to the solution of this problem

$$\log 9{,}260 = 3\ 9666$$
$$\log \sqrt[3]{9{,}260} = \frac{1}{3}(3\ 9666) = 1\ 3222$$
$$\text{antilog } 1\ 3222 = 21\ 0$$

Thus,

$$\sqrt[3]{9{,}260} = 21 \text{ (check } 21 \times 21 \times 21 = 9{,}261)$$

EXAMPLE 14 K

Evaluate $(0\ 0716)^5$ by the process of logarithms

Solution:

Recall that $\log M^n = n \log M$; the application of this concept will provide the solution to the given problem.

$$\log 0.0716 = 8.8549 - 10$$
$$\log (0.0716)^5 = 5 \log 0.0716 = 5(8.8549 - 10)$$
$$= 44.2745 - 50 = 4.2745 - 10$$
$$\text{antilog } 4.2745 - 10 = 0.00000188;$$

Thus,

$$(0.0716)^5 = 1.88 \times 10^{-6}$$

EXERCISES 14-3

Find the numerical values, to three significant figures, of the various expressions, using the process of logarithms.

1. 100×2
2. 20×300
3. $\dfrac{1{,}000}{100}$
4. $\sqrt{100} \times \sqrt{1{,}000}$
5. 7.96×32.4
6. 4.73×0.0261
7. $3.89 \times 10^4 \times 8.62 \times 10^{-3}$
8. $36.2 \times 9.5 \times 102$
9. $\dfrac{3.64 \times 468}{0.52}$
10. $\dfrac{0.099 \times 423}{8.5 \times 0.049}$
11. $\sqrt{169}$
12. $\sqrt[3]{9.26}$
13. $\sqrt{21{,}600}$
14. $\sqrt[3]{0.389}$
15. $\sqrt[4]{0.000207}$
16. $(0.512)^{2/3}$
17. $(23.5)^{1.12}$
18. $(0.375)^{2.40}$
19. $(47.1)^{3/5}$
20. $(700)^{0.35}$

14-6 INTERPOLATION

Table II provides information leading directly to the logarithms of three-digit numbers. By a process called *interpolation*, the table can be extended to include numbers containing an additional digit. The accuracy of the table is not extended, however, only its usage. This procedure associated with interpolation can probably be best explained by working several problems.

EXAMPLE 14-L:

Find the log 26.44.

Solution:

The log 26.44 lies between log 26.40 and log 26.50. It follows that

the mantissa of log 26.44 would then fall between the tabular readings of $N = 2{,}640$ and $N = 2{,}650$. These relationships may, perhaps, appear more meaningful in the form of a diagram (Fig. 14-5).

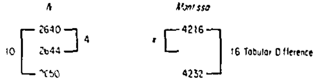


Figure 14-5

The difference between two consecutive readings in the table is called the tabular difference. For the example under discussion, this is $4{,}232 - 4{,}216 = 16$. The difference between the two consecutive numbers, N, corresponding to the mantissas is $2{,}650 - 2{,}640 = 10$. Furthermore, the additional digit is four-tenths of the interval between 2,640 and 2,650. Therefore, it is assumed that the mantissa corresponding to 2,644 is also four-tenths of the difference (interval) between the respective mantissas. It is this *assumption* (actually an approximation) that provides the *basis of interpolation*. Mathematically, this can be represented by the proportion

$$\frac{4}{10} = \frac{x}{16}, \text{ where } x = \frac{4}{10} \times 16 = 6.4 = 6$$

(The figure is rounded off to 6 because a four-place table cannot be extended to a five-place table.)

To complete the computation, 6 is added to 4,216, which provides the required mantissa, $4{,}216 + 6 = 4{,}222$.

Thus, log 26.44 = 1.4222.

The process of finding numbers whose antilogarithms are not listed directly in the table is the inverse of the method just illustrated.

EXAMPLE 14-M.

Find antilog 2.7670.

Solution

By inspection (antilog 2.7670), it is apparent that the characteristic is 2 and the mantissa is 7670. Furthermore, a study of Table II indicates that the mantissa, 7670, lies between the tabular mantissas 7664 and 7672, which correspond to $N = 5840$ and $N = 5850$. Again, a diagram may clarify the relationships (Fig. 14-6).

As previously, based on the assumptions of interpolation,

$$\frac{y}{10} = \frac{6}{8} \text{ where } y = 10 \times \frac{6}{8} = 7.5 \text{ (rounded off)} = 8$$

which is the additional digit. Thus, $N = 5840 + 8 = 5848$ and antilog $2.7670 = 584.8$.

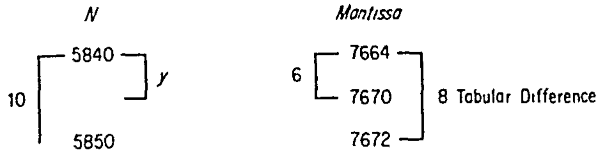


Figure 14-6

REVIEW EXERCISES 14-4

Find the logarithms of the following numbers (Ex. 1-6).

1. 1.003 *2.* 147.4

3. 0.03567 *4.* 0.4215

5. 82.49 *6.* 2.0382×10^{-4}

Find the antilogarithms of the following logarithms (Ex. 7-12).

7. 0.2307 *8.* 1.3587

9. 22.6290 − 20 *10.* 22.6290 − 30

11. 5.88891 *12.* 10.00004

Find the numerical values of the various expressions by the process of logarithms (Ex. 13-18).

13. $\sqrt{131.4}$ *14.* $\sqrt[3]{131.4}$

15. $(2.065)^{1.19}$ *16.* $(0.3648)^{2.20}$

17. $\sqrt{0.9406}$ *18.* $\sqrt[3]{0.005762}$

19. The intensity level of sound can be determined by the formula $b = 10 \log I/I_0$, where b is measured in decibels and I in units of watts per square centimeter. If $b = 100$, and $I_0 = 10^{-16}$, find I.

20. The efficiency, e, of a certain thermal standard is defined by the formula $e = 1 - 1/r^{k-1}$. If $r = 5$ and $k = 1.35$, find e.

21. $N = (100)^{0.8}(7.0)^{0.4}$. Solve for N.

22. If $P_2/P_1 = (V_1/V_2)^k$, solve for P_2 if $P_1 = 14$ lb/in.^{2}, $V_1 = 16.2$ ft^3, $V_2 = 3.0$ ft^3, and $k = 1.3$.

23. If $e = 1 + c - c(P_2/P_1)^{1/n}$ solve for e if $c = 0.10$, $P_2 = 58.8$, $P_1 = 14.7$, and $n = 1.33$.

24. If $P = P_1[2/(n+1]^{n/n-1}$ solve for P if $P_1 = 150$ and $n = 1.6$.

25. Find N if $N = (\log 40)^3$.

26. If $H = k(t_s - t_1)^{1.6}$ solve for H if $k = 2.5$, $t_s = 225$, and $t_1 = 68$.

CHAPTER 15

Geometry

The history and development of mathematics most certainly can be attributed to the constant probing into the many and varying geometric phenomena in the environment

Geometry could have possibly originated as early as 2000 B C because of the need for repeated land surveys necessitated by the annual flooding of the Nile Still a mystery and heralded as one of the world's greatest engineering achievements, the Pyramids were seemingly built according to the principles of geometry

Measurements of distances between celestial bodies, the path of a projectile, the orbit of the earth, gravitational attraction, the lever, , all provided incentive for further and deeper exploration of geometry An early application might be traced to the six spokes of the royal carriage wheel The six-fold division of the circumference of a circle by its radius, probably suggested the sexagismal system, the concept of base 60

From the ancient papyrus, the original engineering handbook, have come the expanded mathematical processes that supplied the early theorists with the necessary tools to refine raw approximations One sector of this science, of special concern to the technician, is *geometry*

The study of geometry, starting with crude symbols inscribed in clay, has now grown to the point where it presently includes plane and solid geometry, analytic geometry, non-Euclidean geometry, trigonometry, and analytic solid geometry, all based on construction and substantiated by proof

The industrial technician can hardly understand his field of specialization without relating some phase of it to geometry In electronics, radar antennas, vector diagrams of circuits, frequency and amplitude as they relate to power, and voltage and current are but a few electrical concepts that provide meaningful design based upon geometric considerations The atomic structure of elements, the force of a gear tooth, the transit and topography,

and the design of containers are further examples of the application of geometry as it relates to the field of technology.

Geometry basically deals with the measurement of lines, angles, areas, surfaces, volumes, and the properties of various figures. Our concern here will be primarily to explore certain fundamental topics of geometry, that may lead to a fuller appreciation of the subject matter associated with the technician's field of specialization. Axioms, postulates, corollaries, and propositions affiliated with geometry will not be re-explored but rather will be re-defined in association with various physical properties. It should be understood that certain principles must be accepted as true on the basis that they are self-evident.

15-1 FUNDAMENTAL CONCEPTS OF GEOMETRY

Most of the material in this section will deal with the fundamental properties and principles of geometry. The statements and definitions supported by geometric construction are, by and large, considered the **postulates** of plane geometry. *A postulate is a statement accepted as true and self-evident.* Some of the statements, on the other hand, will be theorems. *A theorem is a statement or proposition that can be proved.*

The sketches associated with the elements of construction will be made with the straight-edge, scale, protractor, dividers, compass, or a combination of these instruments. The rudiments of drawing are not of primary concern. This technique is used to demonstrate or help visualize a concept. Sketches or rough drawings of pertinent relationships may often suggest a direction leading eventually to the completion of a particular problem. Accurate drawings, on the other hand, may very well serve as an independent check of an algebraic solution. It must be stressed, however, that the most deliberate drawings, at best, are only approximations of precise symbols and should not be scaled as a substitute for analytical justification.

15-1a POINT: A geometric symbol, represented by a dot (.), used to define a location such as a surveyor's mark or the center of curvature of a concave mirror. A point ($P(3, 5)$) has no size or shape.

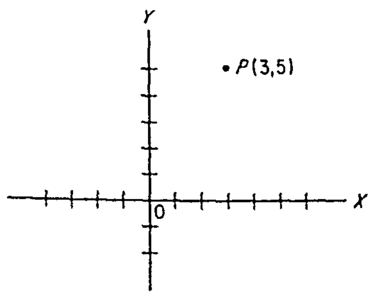


Figure 15-1

15-1b STRAIGHT LINE: The shortest distance between two points, A-B, referred to simply as a line. A line may be extended in either direction

without limit, once it is defined A line may be defined by two points or by one point and a direction, such as *azimuths* and *bearings* The portion of line l, designated by measurement A-B, is called a *line segment* A line can have length and direction but is without thickness or depth

A point may be defined by the intersection of two lines

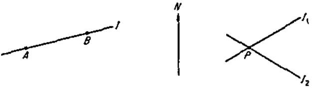


Figure 15-2

15-1c Two *straight lines* can intersect at one point only

15-1d Broken Line An open geometric figure made up of several line segments

15-1e Curved Line A path between two points other than a straight line Unless otherwise indicated, reference to a line will be understood to mean a straight line or line segment

15-1f Parallel Lines Two lines are parallel if they lie in the same plane and do not intersect, no matter how far extended The symbol for parallel lines is $\parallel$

15-1g Only *one straight line* can be drawn between two points

15-1h Plane A flat surface represents a plane This could be compared or likened to the face of a sheet of glass, a chalk board, or the cover of this book

Mathematically, a plane is defined as a surface such that a straight line containing any two points on the surface will be entirely within the surface

15-1i Angle When two lines meet at a point, they form an *angle* The common point is called the *vertex* (Fig 15-3) The symbol for angle is $\angle$ An angle can be designated by its vertex, the end points and the vertex, or by any arbitrary or convenient notation such as $\angle O$, $\angle\alpha$, or $\angle AOB$ (All define the same angle Fig 15-3)

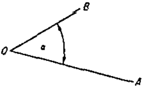


Figure 15-3

One side of an angle is called the *initial side* and the other is called the *terminal side* An angle is usually designated in a counterclockwise direction Furthermore, the vertex (Fig 15-4) is really common to two angles, an *interior angle* and an *exterior angle* Thus, the angle designation must be rather exacting, since the two angles are hardly ever the same measurement

15-1j Measurement of Angles The *size of an angle* is determined

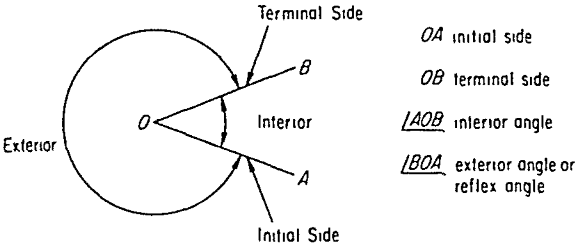


Figure 15-4

by the *difference in direction* of the initial side and the terminal side, or the amount of rotation necessary to bring the initial side coincident with the terminal side.

A complete rotation of a line around a point generates an angle of 360 degrees, or using the notation for degrees, 360° (Fig. 15-5).

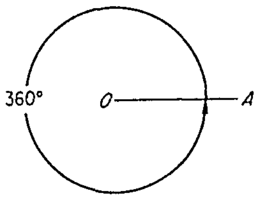


Figure 15-5

One unit of angular measurement is the *degree*, which is $\frac{1}{360}$ of the angle generated by a complete revolution. At one time it was thought that a year contained 360 days. The Babylonians divided a circle into 360 equal parts and each part was to have designated one day.

A degree may be divided into 60 equal smaller angles called *minutes*, indicated as ($'$), where $1° = 60'$. The minute can be further split into 60 equal parts called *seconds* ($''$), where $1' = 60''$ and $1° = 3{,}600''$. Notice that in dealing with degrees the base is 60. One-half degree would be equal to $30'$, $\frac{1}{2}° = 30'$, $\frac{1}{3}° = 20'$, $\frac{1}{6}° = 10'$, ...

In engineering work, angular measurements smaller than $5'$ are seldom used. Some angles may be constructed with accuracy and most often angles are laid off and measured by an instrument called a protractor (Fig. 15-6). Protractors vary in quality and accuracy. Precision protractors, with vernier attachments, can measure angles within $5'$.

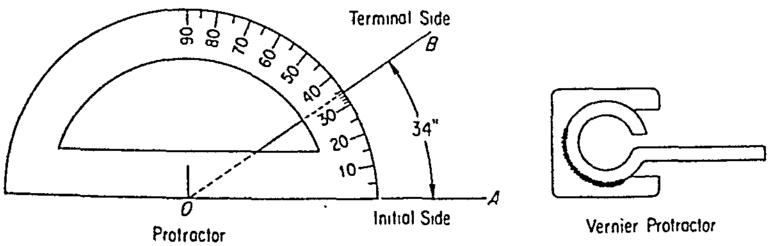


Figure 15-6

An angle that measures 32 degrees, 40 minutes, and 20 seconds is written $32°40'20''$. Again, the technician will seldom have cause for such extreme accuracy simply because the tools of production render this need impractical.

EXAMPLE 15 A

To measure an angle with a protractor (Fig 15 6)

1 Place the center of the protractor at the vertex, O, of the angle to be measured, such as $\angle AOB$

2 Extend the terminal side of angle OB in this illustration so that it falls beyond the scale of the protractor

3 Read the protractor, starting with the scale that begins with zero and proceeding counter-clockwise until a designation (mark) is reached on the scale that coincides with the terminal side of the angle The measurement of $\angle AOB = 34°$

15-1k STRAIGHT ANGLE If the sides of an angle form a straight line, the angle thus formed is called a *straight angle* and measures 180°

15-1l RIGHT ANGLE One half of a straight angle is called a *right angle*, BOC and COA are both right angles and equal to 90° (Fig 15-7) The notation for a right angle is a small square $\llcorner$ When two lines form a right angle, they are referred to as being perpendicular and are indicated by the symbol $\perp$

15-1m An *acute angle* is less than 90°, whereas an *obtuse angle* measures more than 90° but less than 180° (Fig 15-7)

15-1n COMPLEMENTARY ANGLES When the sum of two adjacent angles is equal to 90°, the angles are *complementary*, $\angle BOA$ is complementary to $\angle AOC$ (Fig 15 8)

15-1o SUPPLEMENTARY ANGLES When the sum of two adjacent angles is 180°, the angles are said to be *supplementary* to each other, $\angle BOC$ is supplementary to $\angle COA$ (Fig 15-8)

15-1p BISECTORS The *bisector* of a line is a point or a line that divides it into equal parts

Point O, Fig 15 9, is the bisector of line AB, thus $AO = OB$ The line CO is the bisector of line AB, where $AO = OB$ If CO is perpendicular to AB, then it is called the perpendicular bisector of the line (Fig 15 9)

A line can have only one mid point

15-1q The *bisector* of an angle is a line that divides a given angle into two equal angles An angle can have only one bisector (Fig 15-10)

15-1r The distance from a point to a line is measured along the perpendicular from the point to the line The perpendicular is the shortest distance from a point to a line

15-1s Every point on the bisector of an angle is equidistant from the sides of the bisected angle (Fig 15-10) $P_1B_1 = P_1A_1, P_2B_2 = P_2A_2,$

15-1t Two intersecting lines form *vertical angles* The vertical or opposite angles are equal (Fig 15-11)

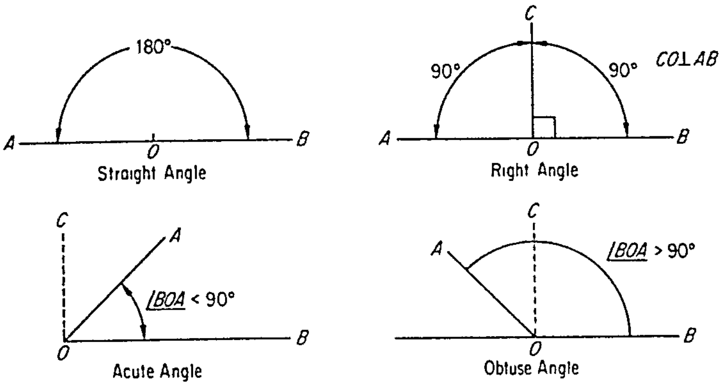


Figure 15-7

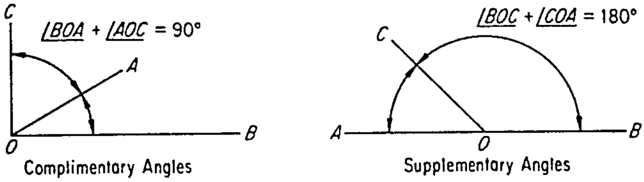


Figure 15-8

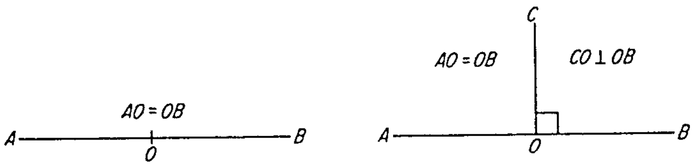


Figure 15-9

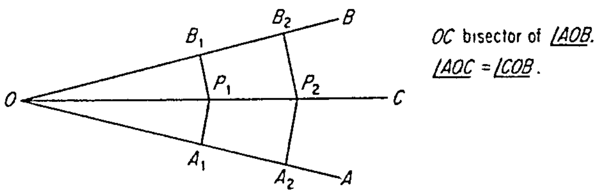


Figure 15-10

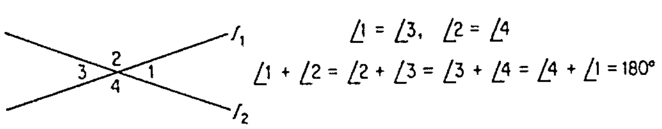


Figure 15-11

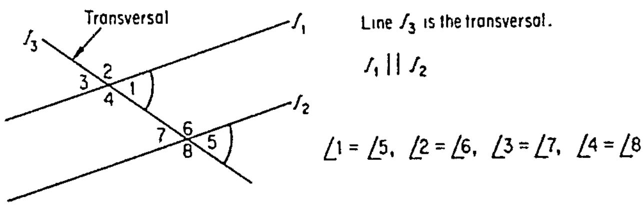


Figure 15-12

Vertical angles appear in pairs; $\angle 1$ and $\angle 3$, as well as $\angle 2$ and $\angle 4$, are vertical or opposite angles.

15-1u TRANSVERSALS: If two lines in a plane are cut by a third line, the third line is called a *transversal* (Fig. 15-12).

15-1v If two parallel lines are cut by a transversal, the corresponding angles are equal (Fig 15-12)

These sets of equal angles represent corresponding angles Corresponding angles are identically positioned with reference to l_1, l_2 and l_3, such as $\underline{/1}$ and $\underline{/5}$ Furthermore, these angles are also on the same side of the transversal

15-1w If two parallel lines are cut by a transversal, the alternate interior angles are equal (Fig 15-13)

15-1x If two parallel lines are cut by a transversal, the alternate exterior angles are equal (Fig 15-14)

15-1y If two lines are cut by a transversal such that two *corresponding angles* are *equal*, the *lines* are *parallel* (Fig 15-15)

15-1z If two lines are cut by a transversal such that two *alternate interior angles* are *equal*, the *lines* are *parallel* (Fig 15-16)

15-1aa If two lines are cut by a transversal such that two *alternate exterior angles* are *equal*, the lines are *parallel* (Fig 15-17)

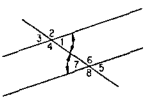


Figure 15-13

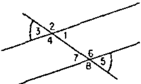


Figure 15-14

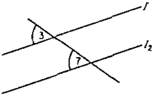


Figure 15-15

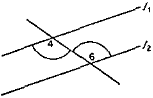


Figure 15-16

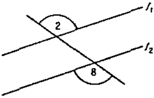


Figure 15-17

15-1ab The distance between two parallel lines is the measure of the common perpendicular (Fig. 15-18).

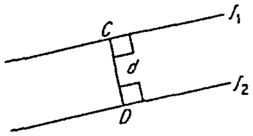


Figure 15-18

15-1ac If several parallel lines cut off equal segments on one transversal, they cut off equal segments on any other transversal (Fig. 15-19). The parallel lines, l_1, l_2, l_3, l_4, l_5 cut transversals l_6 and l_7.

$$\text{Also, } a = b = c = d$$

Conclusion: $a_1 = b_1 = c_1 = d_1$. Notice, however, that this does not indicate that

$$a = a_1, b = b_1, \ldots .$$

EXAMPLE 15-B:

Construct a perpendicular bisector of a line given line AB (Fig. 15-20).

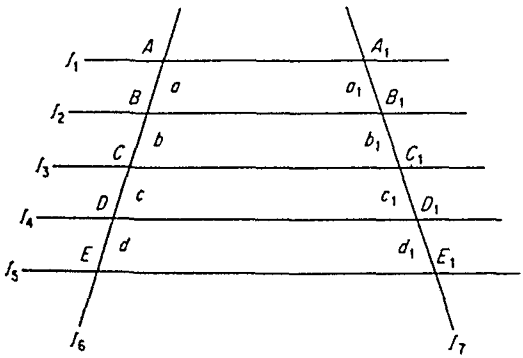


Figure 15-19

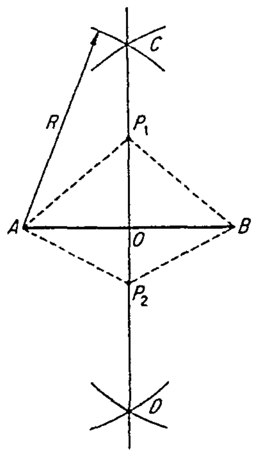


Figure 15-20

Solution:

Step 1. With A as a center and radius, R, equal to AB, construct an arc on both sides of line AB.

Step 2. With B as a center and the same radius, R, construct an arc that intersects the first two arcs at points C and D.

Step 3. Draw line CD. CD will be perpendicular to and bisect line AB at point O. O is the mid-point of AB; thus, $AO = OB$.

Furthermore, every point on the bisector CD is equidistant from the end points AB. ($P_1A = P_1B$; $P_2A = P_2B, \ldots$)

EXAMPLE 15 C

Construct the bisector of an angle given $\angle AOB$ (Fig 15 21)

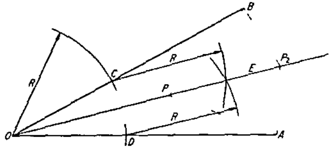


Figure 15-21

Solution

Step 1 With vertex O as a center and any convenient radius R, draw an arc cutting the sides of the angle at C and D respectively

Step 2 With the same radius R, and C and D as centers, respectively, strike off arcs that will intersect inside the angle $\angle AOC$ at E

Step 3 Draw OE The line OE becomes the bisector of the given angle and $\angle AOE = \angle EOB$

Every point on the bisector OE of $\angle AOB$ is equidistant from the end points A B of the angle, thus, $P_1B = P_1A$, $\overline{P_2B} = P_2A$,

EXAMPLE 15 D

Construct an angle equal to a given angle, given $\angle AOB$ and O_1, the vertex of the second angle (Fig 15 22)

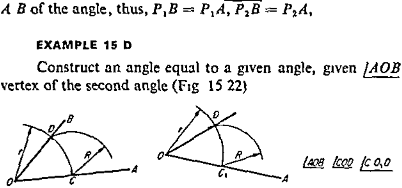


Figure 15 22

Solution

Step 1 With O_1 as an end point, draw any line O_1A_1

Step 2 With O as a center and a convenient radius r, strike off an arc intersecting the sides of $\angle AOB$ at points C and D, respectively

Step 3 With the same radius r and O_1 as a center, construct an arc that cuts across O_1A_1 at point C_1 and extends beyond O_1A_1

Step 4 Using C as a center, set the compass to a radius R equal in length to the line CD

Step 5 With C_1 as a center and radius R, strike an arc intersecting the arc of step 3, at point D_1

Step 6 O_1D_1 is the terminal side of $\angle C_1O_1D_1$ and $\angle C_1O_1D_1$ is equal to angle $\angle AOB$

EXAMPLE 15-E:

Divide a given line into several equal parts.
Divide line $AB = 7\frac{7}{8}$ in. into five equal segments (Fig. 15-23).

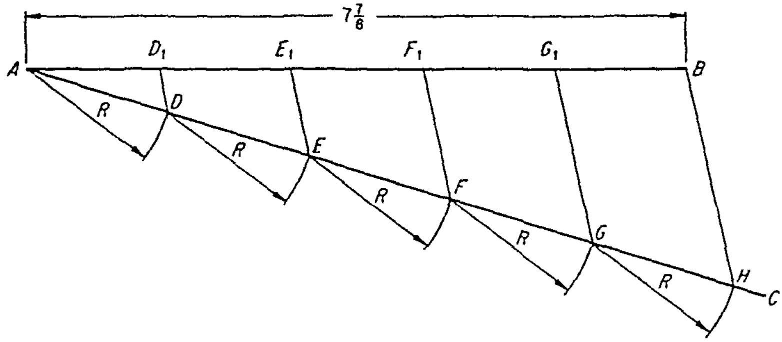


Figure 15-23

Solution:

Step 1. From point A, draw a line AC at any convenient angle, directed toward B.

Step 2. With a suitable radius R and A as a center, strike off an arc cutting AC at D.

Step 3. With D as a center and the same radius R, cut line AC at E. Continue this procedure until the desired number of divisions (segments) are obtained. Thus, $AD = DE = EF = FG = GH$.

Step 4. Connect end points H and B with line HB.

Step 5. Construct lines through D, E, F, and G parallel to BH and cutting AB at D_1, E_1, F_1, and G_1, respectively. Points D_1, E_1, F_1, and G_1 are the required points of division from which $AD_1 = D_1E_1 = E_1F_1 = F_1G_1 = G_1B = AB/5 = 7\frac{7}{8}/5$.

Notice that this is really an application of 15-1ac.

EXERCISES 15-1

The tools of construction involve the scale (ruler), compass, and straight-edge (triangle) only.

1. Construct a line parallel to another line 3 in. apart.

2. Construct a line containing point P (5, 0) and parallel to a line passing through P (0, 0) and P (3, 5).

3. Measure the angle formed by sides going through P_1 (0, 8), P_2 (0, 0), and P_3 (7, 7). (Use the protractor.)

4. Construct an angle of 67°30′.

5. Construct a line containing point P (5, 0) and perpendicular to a line passing through P (0, 0) and P (3, 5).

6. Find, by construction, the mid-point of a line determined by $P(-5, 2)$ and $P(2, -5)$

7. At the point $P(2, 3)$, construct a right angle such that one side will pass through $P(-1, -1)$

8. Divide an inch into nine equal parts

9. Construct an angle of $112°30'$ and bisect it

10. Given two parallel lines, l_1 and l_2, line l_1 passes through $P(0, 8)$ and $P(4, 9)$, whereas l_2 contains $P(0, -2)$ and $P(8, 0)$ Construct a third line parallel to l_1 and l_2 that will be twice the distance from l_1 as it is from l_2

11. Construct a line parallel to the lines in problem 10, equidistant from each of the given lines (l_1 and l_2)

12. Two parallel lines (Fig 15-24), l_1 and l_2, are cut by transversal l_3, forming angles as indicated Find $\angle A$, $\angle B$, $\angle C$, and $\angle D$

13. Indicate which of the angles (Fig 15 25) are (a) corresponding angles, (b) alternate interior, and (c) alternate exterior angles

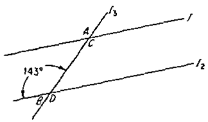


Figure 15-24

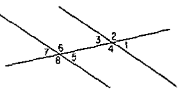


Figure 15-25

14. If l_1 is parallel to l_2, find $\angle A$, $\angle B$, $\angle C$, $\angle D$, and $\angle F$ (Fig 15-26)

15. (a) Construct a line five units away from the origin $P(0, 0)$, which is neither parallel nor perpendicular to the coordinate axis (b) Construct a line parallel to the line of 15-a, ten units on either side of it

16. Construct an angle of 135°

17. Construct a line perpendicular to a given line, AB, from a point P outside the given line (Fig 15-27)

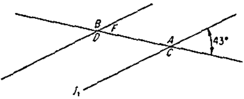


Figure 15-26

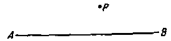


Figure 15-27

18. Construct a perpendicular to a given line, AB, at a given point P on the line (Fig 15-28)

19. Construct a line parallel to a given line, AB, through a point P not on the given line (Fig. 15-29).

20. Given an acute angle AOB, construct a perpendicular from P (located on OB) to side OA (Fig. 15-30).

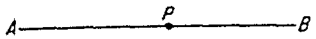


Figure 15-28

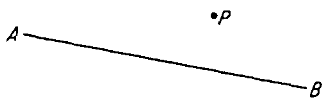


Figure 15-29

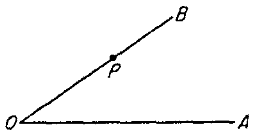


Figure 15-30

15-2 CIRCLES

A *circle* is a closed curve, all points of which are equidistant from a fixed point called the center. The distance from the center to any point on the circle is called the radius. A line passing through the center of the circle, touching the end points of the circle, is called the diameter and is twice the radius. The circumference is the distance around or length of the circle. The ratio of the circumference to the diameter is constant for every circle and is defined by π (pi).

$$\frac{\text{Circumference } (C)}{\text{Diameter } (d)} = \pi = 3.1416 \text{ (approximately)}$$

Furthermore,

$$C = \pi d, \text{ or } C = 2\pi r$$

The area of a circle is given by the formula:

$$A = \frac{\pi d^2}{4} = \pi r^2$$

15-2a An *arc* of a circle is any part of the curve less than the circumference. The notation for arc is $\frown$, such as $\overset{\frown}{AB}$ (Fig. 15-31).

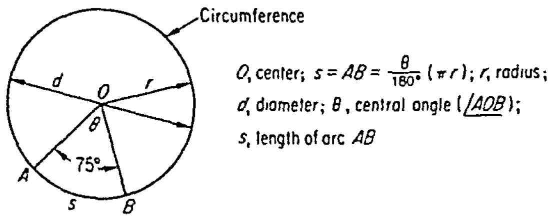


Figure 15-31

15-2b A *central angle* of a circle is any angle with a vertex at the center of the circle. If radii (plural for radius) are drawn from the end points of an arc, they will form a central angle. The central angle has the same measure in degrees as its subtended arc. If the central angle measures 75°, then the

intercepted arc measures 75° If $\theta = \angle AOB = 75°$, it follows that $\widehat{AB} = 75°$ (Fig 15-31)

15-2c The length of an arc of a circle varies jointly with the central angle and the circumference We have $s \propto \theta C$, from which $s = \theta/180°(\pi r)$ (The measure of an arc is an angular measurement corresponding to its central angle and is usually in degrees The length of an arc is a linear measurement and is in the same units as the radius)

15-2d A *chord* of a circle is a line segment whose end points touch the circle If a chord passes through the center of a circle, it becomes a diameter The notation for a chord is —, such as $\overline{CD}$

15-2e A *sector* of a circle is that area included between an arc and the radii that form the central angle (Fig 15-32)

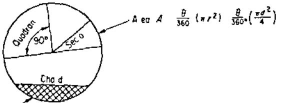


Figure 15-32

15-2f A *segment* of a circle is the area between a chord and the intercepted arc (Fig 15-32)

The area of a sector of a circle is to the area of a circle as the central angle is to 360°

$$\frac{A}{\pi r^2} = \frac{\theta}{360°}, \quad \text{or} \quad A(\text{sector}) = \frac{\theta}{360°}(\pi r^2) = \frac{\theta}{360°}\left(\frac{\pi d^2}{4}\right)$$

15-2g An *inscribed angle* is measured by half of its intercepted arc (Fig 15-33)

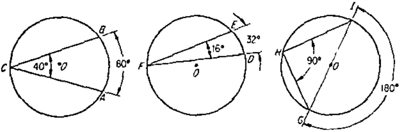


Figure 15-33

An inscribed angle is an angle with its vertex on the circumference, whose sides are chords of the circle $\angle ACB$ and $\angle DFE$ are inscribed angles

15-2h An angle inscribed in a semicircle is a right angle (90°) (Fig 15-33)

15-2i In the same circle, equal central angles subtend equal arcs and equal chords

15-2j A line that touches a circle at one point, P, is called a *tangent*. The point P is called the point of tangency. A line that cuts the circle at two points, A, B, and passes through it, is called a *secant* (Fig. 15-34).

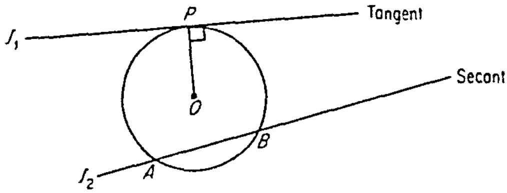


Figure 15-34

15-2k A radius drawn to the point of tangency is perpendicular to the tangent. $OP \perp l_1$ (Fig. 15-34).

15-2l The *two tangents* drawn to a circle from an outside point are equal. Furthermore, the angles formed by the tangents and a line from the point to the center of the given circle are equal. $AP = BP$ and $\angle APO = \angle OPB$ (Fig. 15-35).

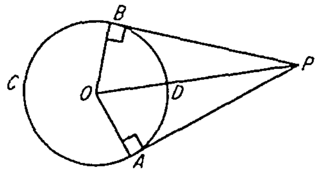


Figure 15-35

Angle APB, formed by the tangents, with vertex at P, is equal to one half the difference of the intercepted arcs.

$\angle APB = (\overset{\frown}{BCA} - \overset{\frown}{ADB})/2$, where $\overset{\frown}{BCA}$ is called the *major arc* and $\overset{\frown}{ADB}$, the *minor arc*.

If $\overset{\frown}{ADB} = 142°$ and $\overset{\frown}{BCA} = 218°$, then $\angle APB = (218° - 142°)/2 = 76°/2 = 38°$.

15-2m An angle formed by the intersection of two chords is equal to one half the sum of the measure of the two intercepted arcs. $\angle APB = \angle CPD = \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD})$ (Fig. 15-36).

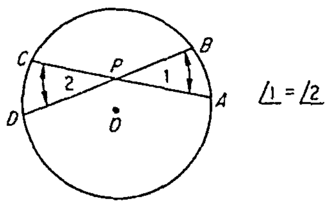
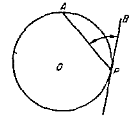


Figure 15-36

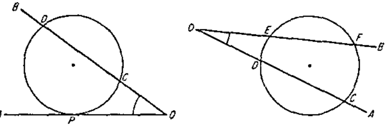
If $\overset{\frown}{AB} = 36°$ and $\overset{\frown}{CD} = 82°$, then $\angle APB = \angle CPD = \frac{1}{2}(36° + 82°) = 118°/2 = 59°$.

15-2n An angle formed by the intersection of a chord and a tangent (at the point of tangency) is measured by one half the intercepted arc (Fig. 15-37).



$\angle BPA = \frac{1}{2}(\widehat{PA})$
If $\widehat{PA} = 75°$, $\angle BPA = \frac{1}{2}(75°) = 37.5° = 37°30'$

Figure 15-37



(a) (b) Figure 15-38

15-2o An angle formed outside the circle, by the intersection of a secant and a tangent or two secants, is equal to one half the difference of the measure of the two intercepted arcs (Fig 15-38)

(a) $\angle AOB = \frac{1}{2}(\widehat{PD} - \widehat{PC})$

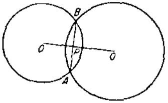
If $\widehat{PD} = 162°$ and $\widehat{PC} = 94°$, $\angle AOP = \frac{1}{2}(162° - 94°) = 68/2° = 34°$

(b) $\angle AOB = \frac{1}{2}(\widehat{CF} - \widehat{DE})$

If $\widehat{CF} = 102°$ and $\widehat{DE} = 48°$, $\angle AOB = \frac{1}{2}(102° - 48°) = 54°/2 = 27°$

15-2p A radius, or diameter, perpendicular to a chord bisects the chord, its subtended arc, and the central angle

15-2q If two circles intersect, their line of centers will be the perpendicular bisector of the common chord (Fig 15 39)



Line OO is perpendicular bisector of chord AB $OO \perp AB$ $AP = PB$

Figure 15-39

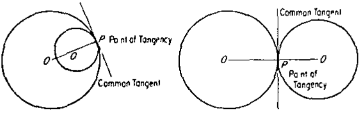


Figure 15-40

15-2r If two circles are tangent to each other, their line of centers will pass through the point of tangency and be perpendicular to the common tangent (Fig 15-40)

In Fig. 15-40a the circles are tangent internally.
In Fig. 15-40b, the circles are tangent externally.

15-2s Through three points not on a straight line, one and only one circle can be constructed.

EXAMPLE 15-F:

Construct a circle through three given points, not on a straight line. Given P_1, P_2, and P_3 not on a straight line (Fig. 15-41).

Solution:

Draw line segments P_1P_2 and P_2P_3 and construct the perpendicular bisector l_1 of P_1P_2 and l_2 of P_2P_3. Extend l_1 and l_2 until they intersect inside P_1, P_2, and P_3 at point O. (Recall: Radius is perpendicular bisector of chord.) Thus, the point of intersection of the perpendicular bisectors is the center of the circle of construction. Furthermore, OP_1, OP_2, and OP_3 are radii of the circle determined by the three given points. With O as the center and OP_1 (or OP_2 or OP_3) as the radius, complete the construction.

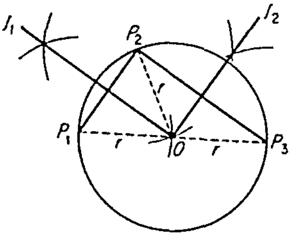


Figure 15-41

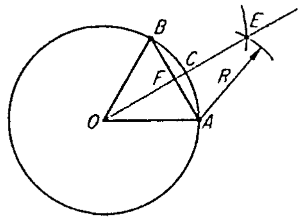


Figure 15-42

EXAMPLE 15-G:

Construct the bisector of an arc, given a circle, O, with arc AB (Fig. 15-42).

Solution:

Draw radii OA and OB. Then with A and B as centers and radius R, strike off arcs outside circle O intersecting at E. The points O and E thus determine the bisector of arc AB (which intersects the circle at C). From the construction, it is further evident that:

(a) $OC \perp \overline{AB}$ and $AF = FB = \frac{\overline{AB}}{2}$.

and

(b) OC bisects central angle AOB as well as $\widehat{AB}$.

Therefore,

$$\underline{/AOC} = \underline{/COB} = \frac{\underline{/AOB}}{2} \text{ and } \widehat{AC} = \widehat{CB} = \frac{\widehat{AB}}{2}$$

EXAMPLE 15 H

Divide a circle into six equal arcs and six equal central angles (Fig 15 43)

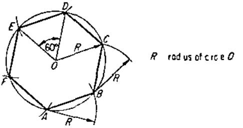


Figure 15-43

Solution

Starting at A, if successive arcs with radius R (radius of given circle) are inscribed on the circumference, with centers at the intersection of the previous arc, six equal arcs will be formed The measure of the arc and its central angle will be equal to 60° Furthermore, six equal sectors, AOB, BOC, COD, , will be developed with the construction of the radii to the divisions on the circumference This provides a basis for constructing various angles that are at times useful in the field

Starting with the central angle of 60°, successive bisects (bisecting the bisected angle) lead to the construction of angles of 30°, 15°, $7\frac{1}{2}$°, , and with a little application, angles of 45°, 75°, 105°, 165°,

Chords drawn to the divisions on the circumference develop a geometric figure called a regular hexagon, in which each side is equal to the radius of the given circle

EXERCISES 15-2

The only instruments needed for construction problems are a compass and straight-edge

1. Construct a tangent to a circle at a point on the circle (any convenient radius)

2. Construct a circle containing these three points $P_1(2, -2)$, $P_2(3, 3)$, and $P_3(8, 2)$

3. Given a chord of a circle with end points $P_1(3, 4)$ and $P_2(4, -3)$, construct a circle containing the given chord

4. Given a circle with a radius equal to 5 units, draw two chords within this circle each 7 units long Compare their distances from the center

5. Given a circle of radius 5 units, construct two chords within this circle 3 units away from the center and compare their lengths (Choice of units is left to the student This could be an inch, a centimeter, one half inch, etc)

6. Make a generalization (if one exists) suggested by exercises 4 and 5

7. Construct the following angles (check with protractor).

(a)	$120°$	(g)	$11\frac{1}{4}°$	(m)	$18°45'$
(b)	$165°$	(h)	$11°15'$	(n)	$13\frac{1}{8}°$
(c)	$15°$	(i)	$7°30'$	(o)	$105°$
(d)	$30°$	(j)	$52\frac{1}{2}°$	(p)	$82\frac{1}{2}°$
(e)	$90°$	(k)	$26\frac{1}{4}°$	(q)	$210°$
(f)	$22\frac{1}{2}°$	(l)	$67°30'$	(r)	$300°$

8. The angle formed by two tangents from a common point outside the circle is 30°. The length of the tangent is 6 units. Construct the circle that meets these conditions.

9. A chord with a length of 4 units has a central angle of 75°. Construct the circle defined by these conditions.

10. A central angle and an inscribed angle have a common arc of 62°, in the same circle. Find the measure of the central angle and the inscribed angle.

11. Two chords intersecting inside a circle intercept arcs that measure 67° and 93°, respectively. Find the vertical angles of the intersecting chords.

12. Two secants intersecting outside a circle cut arcs of 108° and 74°, respectively. Find the angle formed by the secants.

13. At the point of tangency, a chord and a tangent form an angle of 60°. Construct a circle defined by the angle if the length of the chord is 3 units. Find the measure of the intercepted arc and the subtended central angle.

14. A circle with a radius equal to 10 cm has a central angle of 120°. Find the area of the sector and the length of the intercepted arc.

15. Find the area of circle O if $AB = 72°$ and $s = 12$ in. (Fig. 15-44.)

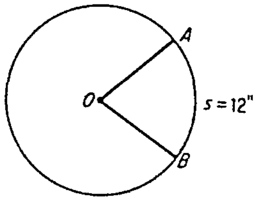


Figure 15-44

16. If AP and BP are tangent to circle O, $\overset{\frown}{ACB} = 235°$, find the measure of the minor arc AB and $\angle BPA$.

17. Given circle O and $\overset{\frown}{EF} + \overset{\frown}{CD} = 120°$, find $\angle BPA$, if $\overset{\frown}{CD} = 3\ \overset{\frown}{EF}$.

18. Find $\angle BPA$ if $\overset{\frown}{ACB} = 2\ \overset{\frown}{BA}$ (Fig. 15-45).

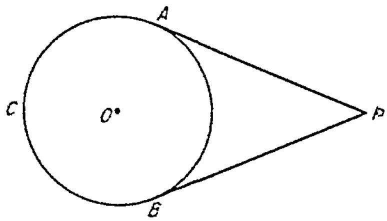


Figure 15-45

19. Find $\widehat{CD}$ if $\widehat{EF} = \frac{1}{4}\widehat{CD}$ and $\angle BPA = 39°$ (Fig 15-46)

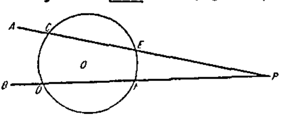


Figure 15-46

20. A circle with a radius of 5 in has a central angle that intercepts an arc whose length is 5 in Find the area of this sector

21. The center distances of two tangent circles is 12 units Construct these circles if the radius of one is twice the radius of the other

22. Construct two intersecting circles whose common chord is 6 units The radius of the first circle is 6 units and the radius of the second is 9 units

23 Find the area of a sector of a circle whose central angle is equal to $\pi°/2$ and whose diameter is equal to 20 in

24. O is the center of the circle If $\widehat{CD} = 125°$, find $\angle BOC$, $\angle AOD$, $\widehat{AD}$, $\angle BDC$, and $\angle ACD$

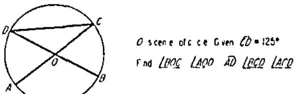


Figure 15-47

25. O is the center of the circle If $\widehat{CD} = 80°$, $\angle ABC = 45°$, $\widehat{AB} = 100°$, find $\angle BCD$ and $\angle AOC$

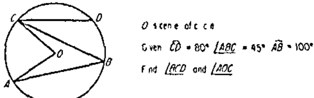


Figure 15-48

26. O is the center of the circle, PA is a tangent, and PC is a secant If $\widehat{AB} = 110°$, and $\angle APB = 30°$, find $\angle BAC$, $\angle BOC$, and $\angle BCA$

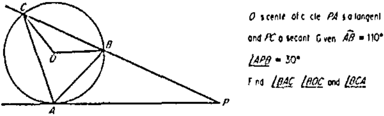


Figure 15-49

27. Construct a circle that is tangent to a line segment whose end points are P_1 (0, 0) and P_2 (10, 4). The radius of the circle is equal to one half the length of the line segment. The point of tangency is at the mid-point of the line (2 solutions).

28. Construct a circle going through the points P_1 (−5, −5) and P_2 (5, 0), with its center 7 units away from P (0, 0) (2 solutions).

15-3 POLYGONS

A *polygon* is defined as a closed broken line in a plane. The first and last (end) points are coincident (same).
The geometric forms in Figure 15-51 represent several common polygons.

15-3a TRIANGLE: A *triangle* is a closed plane figure with three sides and three angles. The sum of the angles of a triangle is 180° (Fig. 15-52).

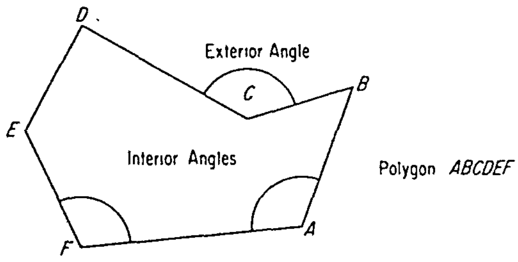


Figure 15-50

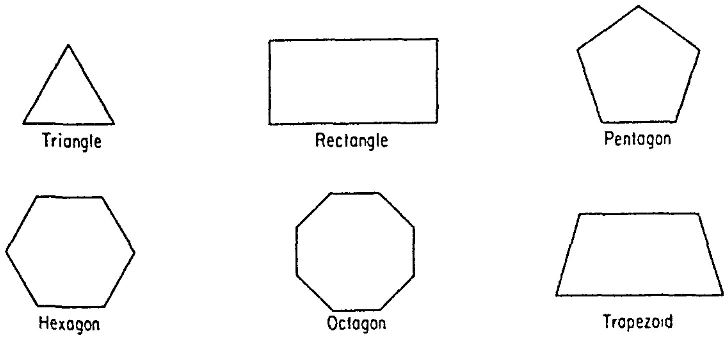


Figure 15-51

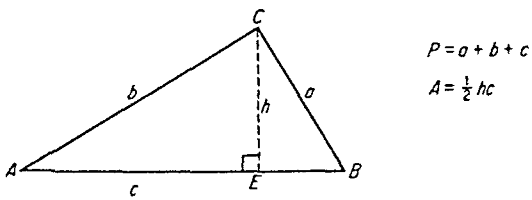


Figure 15-52

A triangle is defined by the symbol $\triangle$, along with the notation of the vertices, such as $\triangle ABC$. Capital letters usually define the vertices, whereas the corresponding lower-case letters represent the sides opposite the respective angles. The sides of $\triangle ABC$ are a, b, and c. A perpendicular from a vertex to the opposite side of a triangle is called an *altitude*, h, where CE (h) is perpen-

dicular to AB Every triangle can have three altitudes, although only one is involved

The *perimeter* of a triangle is equal to the sum of its sides

$$p = a + b + c$$

The *area* of a triangle is equal to one half the product of the altitude and base, wherein the side containing the altitude becomes the base, such as AB (Fig 15-52)

$$\text{Area } A = \frac{1}{2}(\text{base})(\text{altitude}) = \frac{1}{2}hc$$

A more complicated formula has been developed for finding the area of a triangle involving all the sides

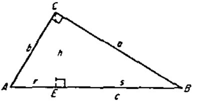
$$A = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a + b + c)$$

or

$$s = \frac{1}{2}p \ (p \text{ is the perimeter of } \triangle)$$

The triangles are classified according to sides and angles

15-3b Right Triangle A *right triangle* contains one right angle (90°) and two acute angles The side opposite the right angle is the longest side of this triangle and is called the *hypotenuse*, whereas the other two sides are referred to as the *legs* In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides Furthermore, an altitude from the vertex containing the right angle divides the triangle into two right triangles, each of whose angles are equal to the angles of the original triangle (Fig 15 53)



$c^2 \quad a^2 + b^2 \quad h^2 \quad b^2 \quad r^2 \quad a^2 \quad s^2$

$r \quad \frac{h}{s} \quad A \quad {}_2 hc$

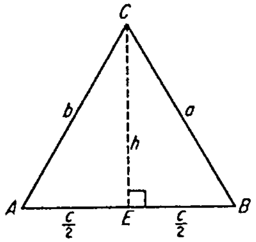
Figure 15 53

The altitude to the hypotenuse is the *mean proportional* between the segments of the hypotenuse

15-3c Isosceles Triangle An *isosceles triangle* contains two sides that are equal in length The angle formed by the equal sides is called the vertex whereas the angles opposite the equal sides are called the base angles contained by a common side called the base The *base angles* of an isosceles triangle are equal An altitude from the vertex to the base divides the isosceles triangle into two *congruent right* triangles

Congruent triangles (figures) are triangles whose corresponding sides and angles are equal in measurement.

Thus, an altitude dropped from the vertex angle bisects the vertex angle and the base. Furthermore, the altitude, h, is a perpendicular bisector of AB, where $AE = EB = AB/2$ or $c/2$ (Fig. 15-54).



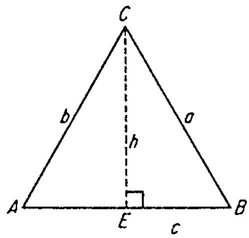
$AC = BC$ or $b = a$, $\angle A = \angle B$ h, altitude;

$AE = EB$; $h^2 = b^2 - (\frac{c}{2})^2 = a^2 - (\frac{c}{2})^2$;

$A = \frac{1}{2}hc$

Figure 15-54

15-3d Equilateral Triangle: An *equilateral triangle* is a triangle whose three sides are equal. An *equilateral triangle* is also an *equiangular triangle*; all three angles are equal. An equilateral triangle can also be classified as a special case of an isosceles triangle. Most of the conditions associated with an isosceles triangle are applicable to an equilateral triangle (Fig. 15-55).



$\angle A = \angle B = \angle C = 60°$

$AC = AB = BC$, or, $a = b = c$

Figure 15-55

15-3e A *scalene triangle* has no equal sides nor equal angles.

15-3f An *acute triangle* has all angles less than 90°.

15-3g An *obtuse triangle* has one angle greater than 90°.

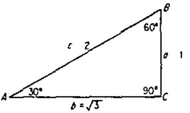
These terms, 15-3e, 3f, 3g, are seldom used, however. If a triangle is neither a right, isosceles, nor equilateral, it is referred to simply as a triangle, or more appropriately, a general triangle. Furthermore, if a triangle does not contain a right angle, it is called an *oblique triangle*.

15-3h In any triangle, an exterior angle is equal to the sum of the two opposite interior angles.

15-3i The 30°-60° and 45°, Right Triangles: There are two right triangles that have several unique features among triangles in general. Certain conditions associated with these two particular triangles appear frequently in the fields of engineering, mathematics, and science, especially in the area of drafting.

First of all, the 30°-60°-90° triangle is made up of angles that are multiples of 30. The distinguishing feature, however, is that the side opposite

the 30° angle is one half the hypotenuse, or the hypotenuse is twice the short leg Thus, the sides are to one another as indicated in Fig 15-56

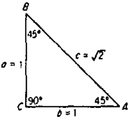


$c^2 = a^2 + b^2$

$(2)^2 = (1)^2 + (\sqrt{3})^2 \quad 4 = 1 + 3$

Figure 15-56

The 45°-right triangle has the distinction of being the one and only isosceles right triangle In an isosceles triangle, two angles are equal and two sides are equal Thus, the sides are to one another as indicated in Fig 15-57



$c^2 = a^2 + b^2$

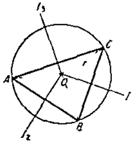
$(\sqrt{2})^2 = (1)^2 + (1)^2 \quad 2 = 1 + 1$

Figure 15-57

15-3j Any *triangle inscribed* in a semicircle is a right triangle

15-3k The *sum* of the lengths of the two shorter sides of a triangle is always greater than the length of the longest side ($a + b > c$)

15-3l The *perpendicular bisectors* of the sides of a triangle meet at a point (are concurrent) that is equidistant from the vertices of the triangle This common point also defines the center of the circumscribing circle (Fig 15-58)



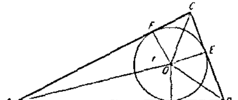
$l_1 \perp$ bisector CB $\quad l_2 \perp$ bisector AB

$l_3 \perp$ bisector AC

$OC = OA = OB$ = radius circle O (r)

Figure 15-58

15-3m The *bisectors of the angles* of a triangle are concurrent at a point equidistant from the sides of the triangle This point of intersection also defines the center of the inscribed circle (Fig 15-59)



OC bisector $\angle C$ $\quad OA$ bisector $\angle A$

OB bisector $\angle B$ $\quad O$ center of

inscribed circle with radius r

Figure 15-59

15-3n The *medians of a triangle* intersect at a common point called the centroid of the triangle. *Medians* are lines drawn from a vertex to the mid-point of the opposite side. *Centroid* is a term used to define the center of gravity or the point of balance for a physical mass. This point of concurrency is two thirds of the distance from a vertex to the mid-point of the opposite side (Fig. 15-60).

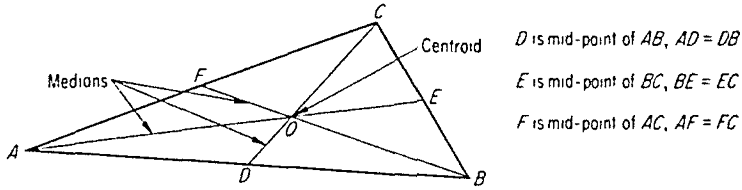
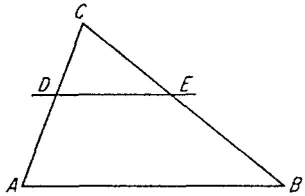


Figure 15-60

15-3o The altitudes of a triangle are concurrent.

15-3p A line parallel to one side of a triangle, extended, divides the other two sides proportionally (Fig. 15-61).



$DE \parallel AB$

$\frac{CD}{DA} = \frac{CE}{EB}$, $\frac{CD}{CA} = \frac{CE}{CB}$

Figure 15-61

15-4 CONGRUENT TRIANGLES

Two or more geometric figures are said to be *congruent* if their corresponding elements or parts (angles, sides, radii, ...) are equal. Corresponding parts of geometric figures occupy the same relative positions within the figures. If congruent figures could be placed coincident, the corresponding elements would then fall on top of one another. The symbol for congruency is $\cong$. With respect to congruent triangles, this means that the three angles are equal and the three corresponding sides are also equal in measurement (Fig. 15-62).

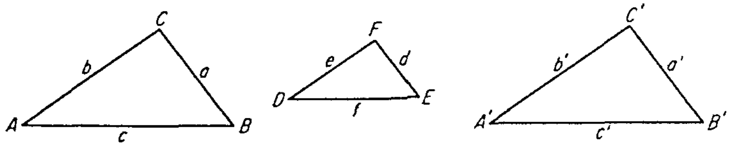


Figure 15-62

Similar triangles, on the other hand, are defined as triangles having equal angles with corresponding sides proportional. These conditions of similarity apply to all geometric figures. The symbol $\sim$ is used to denote similar figures.

If $\triangle ABC \cong \triangle A'B'C'$, it follows that $AB = A'B'$, $BC = B'C'$, $CA = C'A'$, and $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, where AB and $A'B'$, BC and $B'C'$, $\angle C$ and $\angle C'$, . . . , are corresponding parts of the congruent triangles ABC and $A'B'C'$.

If $\triangle ABC \sim \triangle DEF$, it follows that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and $a/d = b/e = c/f$, or $b/c = e/f$, $a/b = d/e$, and $c/a = f/d$

Several conditions that make for congruent triangles are listed below without any attempt at proof These statements are called theorems, and their relationships concerning triangles are tried and true A clear understanding of these concepts will be more meaningful to the technician than memorized, verbatim proofs

Congruent triangles are of great importance to the draftsman, surveyor, and designer The same applies to the application of geometric figures of similarity

15-4a Triangles are congruent if *three sides* of one are equal, respectively, to three sides of the other

15-4b Triangles are congruent if *two angles and the included side* of one are equal, respectively, to two angles and the included side of the other

15-4c Triangles are congruent if *two sides and the included angle* of one are equal, respectively, to two sides and the included angle of the other

15-4d Triangles are congruent if *two angles and a side of one are equal* to two angles and the corresponding side of the other

15-4e *Right triangles* are congruent if a *leg and an acute angle* of one are equal to an acute angle and corresponding leg of the other

15-4f *Right triangles* are congruent if the *hypotenuse and an acute angle* of one are equal, respectively, to the hypotenuse and an acute angle of the other

15-4g *Right triangles* are congruent if *two legs* of one are equal, respectively, to two legs of the other

15-4h *Right triangles* are congruent if the *hypotenuse and leg* of one are equal, respectively, to the hypotenuse and leg of the other

15-5 SIMILAR TRIANGLES

Triangles are *similar* if their *angles are equal*, or the *corresponding sides* are *proportional* Corresponding sides of similar triangles refer to the sides opposite the respective equal angles (Fig 15-63)

15-5a If *two angles* of one triangle are *equal* to two angles of another triangle, the remaining angles are equal and the triangles are *similar*

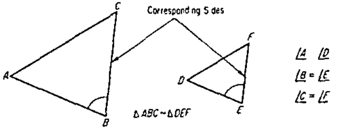


Figure 15-63

15-5b *Right triangles* are *similar* if an *acute angle of one* is equal to an acute angle of the other. This statement, complements 15-5a.

15-5c Triangles are *similar* if their *corresponding sides are in proportion.*

15-5d In any triangle, the altitude drawn to the hypotenuse, constructs two triangles that are similar to each other and to the original triangle, as well.

15-5e The perimeters of two similar triangles are to each other as any two corresponding sides.

15-5f The areas of similar triangles are in the ratio of the squares of any two corresponding sides.

15-5g The corresponding altitudes of similar triangles are to each other as any two corresponding sides.

EXAMPLE 15-1:

Construct a 30°-60° right triangle with its hypotenuse equal to 10 units. Given: Angles and hypotenuse of a right triangle (Fig. 15-64).

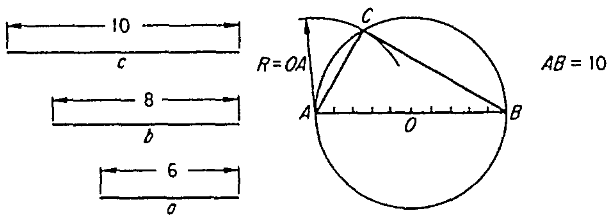


Figure 15-64

Solution:

Lay off the hypotenuse, AB, to a suitable scale representing 10 units and locate the mid-point of AB. This can be done during the initial lay out or by construction (bisect AB).

With O as a center and radius R, where $R = AB/2 = 10/2 = 5$, or $R = AO$ (or OB), draw a circle. This circle will contain the end points A and B. Thus, the hypotenuse of the triangle of construction also becomes the diameter of the immediate circle. Recall that an angle inscribed in a semicircle is a right angle. Therefore, from any point on the circumference (except A or B), lines drawn to A and B, respectively, form a right angle. Furthermore, this angle then becomes the vertex of an inscribed right triangle.

In this illustration, however, the triangle has been defined as a 30°-60° right triangle, which further means that one of the legs must be equal to one half the hypotenuse.

With radius OA, and either A or B as a center, strike off an arc intersecting the semicircle at C. Join C with A and B, respectively. Thus, triangle BCA becomes a 30°-60°-90° triangle, where $C = 90°$ and $AC = AB/2$ (Fig. 15-64).

EXAMPLE 15 J

Construct a triangle with sides 10, 8, and 6 units, respectively

Solution

Lay off one of the given sides to an appropriate scale (In this example the side c is chosen for the demonstration The choice, however, does not affect the construction) Call the end points A and B, respectively

Next, with either A (or B) as center, strike off an arc equal in length to side b Using B as a center and a radius equal to the length of side a, draw an arc, intersecting the first arc at C Draw line segment AC and segment CB Thus, triangle ABC is the triangle of construction with sides 10, 8, and 6, as defined (Fig 15-65)

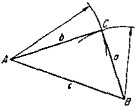


Figure 15-65

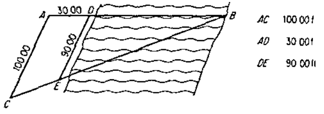


Figure 15 66

EXAMPLE 15 K

In the field of surveying determining inaccessible measurements are frequently made through application of similar triangles

Problem To find the distance between point A and point B without crossing the river (Fig 15 66)

Solution

Locate any accessible point, C, from which a convenient length, $CA = 100\ 00$ ft, can be measured

From point A along a sighted line AB, lay off another convenient distance, such as $AD = 30\ 00$ ft, and from point C lay out a line CB

Next, construct a line through point D parallel to AC, intersecting CB at E, and measure DE ($DE = 90\ 00$ ft)

Thus, $\triangle ABC \sim \triangle DBE$, and it follows that $DB/DE = AB/AC$, where $AB = AD + DB = 30\ 00 + DB$

Substituting accordingly

$$\frac{DB}{90\ 00} = \frac{30\ 00 + DB}{100\ 00},$$

and

$$DB = 90\ 00\left(\frac{30\ 00 + DB}{100\ 00}\right) = \frac{9}{10}(30\ 00 + DB)$$

Furthermore,

$$10\ DB = 270\ 00 + 9\ DB$$

from which

$$DB = 270.00 \text{ ft}$$

Therefore,

$$AB = 30.00 + 270.00 = 300.00 \text{ ft.}$$

EXERCISES 15-3

1. Construct a right triangle with an acute angle equal to 30° and the side opposite the angle equal to 3 units. Find, also, the measure of the remaining sides.

2. Construct a right triangle with legs of 5.00 in. and 12.00 in.

3. Construct a triangle with sides of 4 and 5 units, respectively, and an included angle of 75°.

4. Construct a triangle with sides of 5, 6, and 8 units, respectively, and a similar triangle whose corresponding sides are in the ratio of 2 : 1.

5. Construct the inscribed and circumscribed circles associated with a triangle whose sides measure 4, 5, and 6 units, respectively.

6. Construct an isosceles triangle whose base is 7 units and whose vertex angle measures 45°. (This construction should be completed without the use of a protractor.)

7. Construct the medians of a triangle whose vertices are at the points P_1 (0, 0), P_2 (6, 5), and P_3 (8, −2).

8. Construct the angle bisectors of the triangle in exercise 7.

9. Find the area and perimeter of a triangle defined by points P_1 (0, 0), P_2 (0, 6), and P_3 (−6, 0).

10. Construct the circle that will circumscribe the triangle in exercise 9.

11. Given a triangle with sides that measure 3.00 in., 7.00 in., and 8.00 in., find the perimeter and area of the triangle.

12. Find the area and perimeter of a triangle that is similar to the triangle in exercise 11, whose longest side measures 20.00 in.

13. Construct a 45°-right triangle with a hypotenuse equal to 10 units. Also, find the length of the other two sides.

14. Identify pairs of similar triangles whose corresponding sides are given below:

(a) 25, 7, 24 (b) 5, $3\frac{1}{2}$, 12
(c) 20, 14, 48 (d) $12\frac{1}{2}$, 14, 18
(e) 15, 10, 36 (f) 100, 28, 96

15. For the pairs of similar triangles in exercise 14, determine the ratio of proportionality.

16 Given a triangle with sides of 3 units and 4 units and an included angle of 90°, construct a similar triangle whose longest side measures 15 units

17. Construct an equilateral triangle with sides of 5 units and draw inscribed and circumscribed circles defined by this triangle

18. A right triangle has an acute angle of 60° The altitude to the hypotenuse measures 10 units Construct the triangle and determine the lengths of all three sides Also, find the area

19. Indicate which pair of triangles are similar and which are congruent Give an explanation or reason for your conclusions (Fig 15-67)

For example, triangle *b* and triangle *g* are similar because both are isosceles right triangles

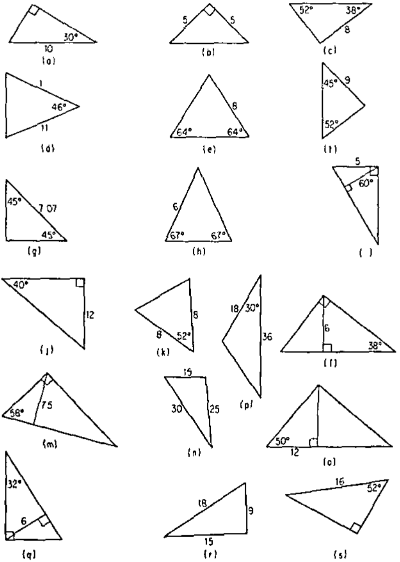


Figure 15-67

20. If $\triangle ABC \cong \triangle DEF$ ($\angle C = \angle F = 90°$), find the area and perimeter of $\triangle ABC$ (Fig 15-68)

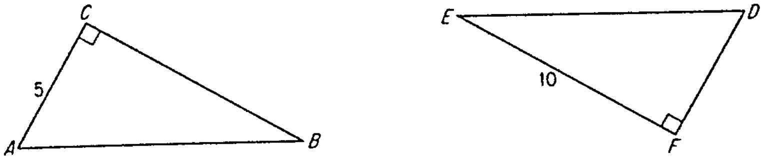


Figure 15-68

21. If $BC \parallel DE$, $BC = 6$, $CE = 3$, and $AE = \frac{4}{3}AC$, find AC and DE (Fig. 15-69).
22. If $DE \parallel AB$, $AB = 90.00$ ft, $DE = 75.00$ ft, and $\angle A = \angle B$, find H (Fig. 15-70).

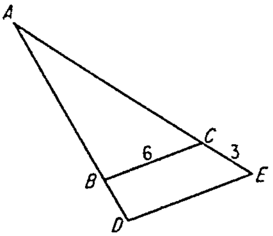


Figure 15-69

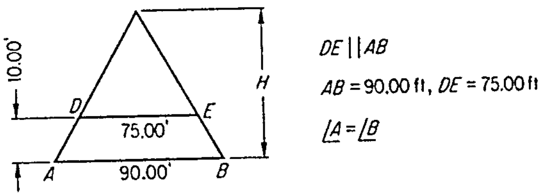


Figure 15-70

15-6 QUADRILATERALS

A *quadrilateral* is a closed plane figure bounded by four straight lines. If the opposite sides of a quadrilateral are parallel, the figure is called a *parallelogram* (Fig. 15-71).

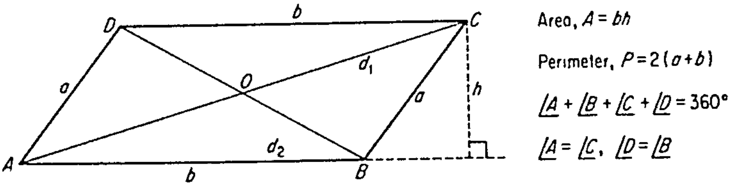


Figure 15-71

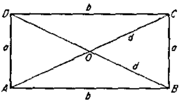
15-6a Properties of a Parallelogram:

1. Opposite sides are equal and parallel.
2. Opposite angles (vertices) are equal.
3. Line segments drawn to opposite vertices are called diagonals, d_1 and d_2. The diagonals of a parallelogram bisect each other. A diagonal also divides the parallelogram into two congruent triangles.

15-6b Rectangle: A *rectangle* is a parallelogram whose interior angles are right angles (Fig. 15-72).

Properties of a rectangle:

1. Opposite sides are equal and parallel.
2. Diagonals of a rectangle are equal and bisect each other.
3. All interior angles are 90°.



$\angle A = \angle B = \angle C = \angle D = 90°$

$A = ab$

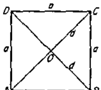
$p = 2(a+b)$

$d = \sqrt{a^2+b^2}$

Figure 15-72

15-6c SQUARE A *square* is a rectangle with equal sides (Fig 15 73)
Properties of a square

1 All sides are equal
2 All interior angles are 90°
3 Diagonals of a square are perpendicular bisectors of each other
4 A diagonal divides a square into two congruent isosceles right triangles (45°-45° 90°)



$\angle A = \angle B = \angle C = \angle D = 90°$

$A = a^2 = \frac{d^2}{2}$

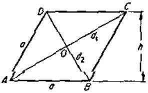
$d = a\sqrt{2}$ or $a = \frac{d\sqrt{2}}{2}$

$p = 4a$

Figure 15-73

15-6d RHOMBUS A *rhombus* is a parallelogram with all sides equal and opposite angles equal (Fig 15-74)
Properties of a rhombus.

1 Opposite sides are equal and parallel
2 Opposite angles (vertices) are equal
3 Diagonals of a rhombus are perpendicular bisectors
4 A diagonal of a rhombus will bisect its angles



$A = \frac{1}{2} d_1 d_2 = ah$

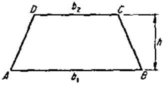
$h = \frac{d_1 d_2}{2a}$

$p = 4a$

$a = \frac{\sqrt{d_1^2 + d_2^2}}{2}$

Figure 15 74

15-6e TRAPEZOID A *trapezoid* is a quadrilateral having two parallel sides (only two) The parallel sides are called the bases (Fig 15-75)



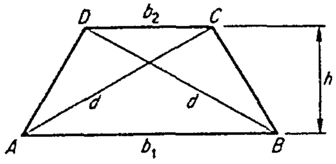
$A = h\left(\frac{b_1+b_2}{2}\right)$

$p = AB + BC + CD + DA$

$AB \parallel DC$

Figure 15 75

15-6f ISOSCELES TRAPEZOID An *isosceles trapezoid* is a trapezoid with non-parallel sides equal (Fig 15-76)



$$A = h\left(\frac{b_1 + b_2}{2}\right), \ \angle A = \angle B$$
$$p = AB + BC + CD + DA$$
$$d = \sqrt{h^2 + \left(\frac{b_1 + b_2}{2}\right)^2}$$

Figure 15-76

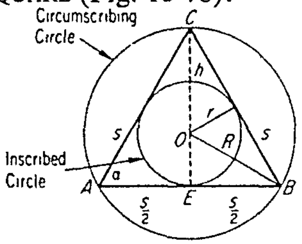
15-7 REGULAR POLYGONS

A *regular polygon* is a closed plane figure with equal sides and equal interior angles. The sum, S, of the interior angles of a regular polygon with n sides is equal to $S = (n - 2)(180°)$. Each interior angle, α, is equal to $\alpha = S/n = (n - 2)(180°)/n$.

A regular polygon can be inscribed in and circumscribed around a circle. The center of the regular polygon is the center for both inscribed and circumscribed circles. Furthermore, the radius, R, of the circumscribed circle is also considered the radius of the regular polygon, whereas the radius, r, of the inscribed circle is called the *apothem*. The area of a regular polygon is $A = \frac{1}{2}rp$, where p is the perimeter.

15-7a EQUILATERAL TRIANGLE (Fig. 15-77):

15-7b SQUARE (Fig. 15-78):



$$p = 3s$$
$$A = \tfrac{1}{2}hs = \tfrac{1}{2}rp$$
$$h = \frac{\sqrt{3}}{2}s$$
$$r = \frac{R}{2}$$
$$s = 1.732R = 3.464r$$
$$\alpha = 60°$$

EQUILATERAL TRIANGLE (3 Sides)

s, Side

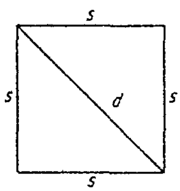
r, Apothem

R, Radius of Polygon

O, Center

h, Altitude

Figure 15-77

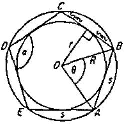


$$p = 4s$$
$$A = \tfrac{1}{2}d^2 = s^2$$
$$d = \sqrt{2}s = 1.414s$$

Figure 15-78 SQUARE (4 sides)

15-7c PENTAGON: In a regular polygon, the apothem, r, is also the perpendicular bisector of the sides (Fig. 15-79).

Sum of interior angles S is $S = (n - 2)180° = 3(180°) = 540°$



$p = 5s$

$A \quad \frac{1}{2} rp = \frac{5}{2} rs$

$s = 1\,176R \quad 1\,452r$

$R^2 \quad r^2 + \left(\frac{s}{2}\right)^2$

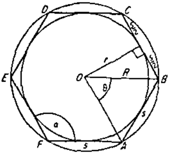
Figure 15 79

Interior angle α is $\alpha = \dfrac{(n-2)}{n}180^\circ = \dfrac{(5-2)}{5}180^\circ = \dfrac{540^\circ}{5} = 108^\circ$

Central angle θ is $\theta = \dfrac{360^\circ}{n} = \dfrac{360^\circ}{5} = 72^\circ$

15-7d HEXAGON (Fig 15 80)

15-7e OCTAGON (Fig 15 81)



$p \quad 6s$

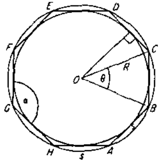
$A = \frac{1}{2} rp \quad 3rs \quad \frac{3\sqrt{3}}{2} s^2$

$R \quad s$

$R^2 \quad r^2 + \left(\frac{s}{2}\right)^2 \quad r \quad \frac{\sqrt{3}}{2} s$

$\theta \quad 60 \quad \alpha \quad 120^\circ \quad S \quad 720^\circ$

Figure 15 80



$p \quad 8s$

$A \quad \frac{1}{2} rp = 4rs$

$R^2 \quad r^2 + \left(\frac{s}{2}\right)^2$

$s \quad 0\,828r \quad 0\,766R$

$\theta \quad 45^\circ \quad S \quad 1080^\circ \quad \alpha \quad 135^\circ$

Figure 15 81

15-7f MISCELLANEOUS REGULAR POLYGONS

15-7g For a given area, the regular polygon has a smaller perimeter than any other polygon with an equivalent number of sides Furthermore the circle, with a circumference equal to the perimeter of any polygon encloses the greater area (Fig 15 83)

Figure 15 82

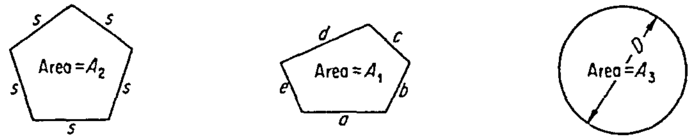


Figure 15-83 REGULAR PENTAGON GENERAL PENTAGON CIRCLE

If $A_1 = A_2$, then $5s < (a + b + c + d + e)$
If $5s = \pi D$, then $A_3 > A_2$
If $\pi D = (a + b + c + d + e)$, then $A_3 > A_1$

EXAMPLE 15-L:

The regular pentagon will be used to develop the formula for the area of a regular polygon (Fig. 15-84).

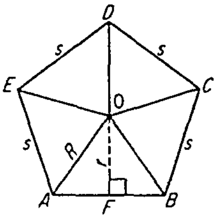


Figure 15-84

Solution:

The perimeter, $p = ns = 5s$, $OF = r$, the apothem (the apothem is the perpendicular bisector of any side).

This indicates that OF or r (the apothem) divides any side into two equal parts and forms an angle of 90° with the side. Furthermore, the apothem bisects the central angle. Thus,

$$AF = FB = \frac{1}{2}s, \text{ and } \angle AOF = \angle FOB = \frac{1}{2}\angle AOB$$

The pentagon (regular polygon) can be divided into five triangles (n triangles for any regular polygon) by joining the vertices with the radius, R. These triangles have equal areas since the altitudes and bases are equal, respectively. Hence,

Area of $\triangle OAB = \frac{1}{2}rs$.

The area of each of the $\triangle$s, BOC, COD, DOE, and EOA, is also equal to $\frac{1}{2}rs$.

Therefore, the area of the pentagon is equal to the sum of the areas of the five triangles, or

$$A = \frac{1}{2}rs + \frac{1}{2}rs + \frac{1}{2}rs + \frac{1}{2}rs + \frac{1}{2}rs = 5\left(\frac{1}{2}rs\right) = 5s\left(\frac{1}{2}r\right),$$

but $5s$ is equal to perimeter, p; thus, $A = \frac{1}{2}rp$.

EXAMPLE 15-M:

Find the area of the sector and segment formed by an arc that measures 120° in a circle whose radius is 8.00 in (Fig. 15-85).

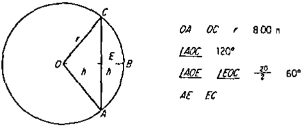


Figure 15 85

Solution

Area of sector $AOC = (120°/360°)\,\pi(8\ 00)^2 = 66\ 97$ in ^{2}

Area of segment ABC = area of sector AOC − area of $\triangle AOC$

Area $\triangle AOC$ $A = \frac{1}{2}h(AC)$

$\angle AOE = 60°$, thus, $\angle OAE = 30°$ and it follows that $h = r/2 = 8\ 00/2 = 4\ 00$ in

$(AE)^2 = (EC)^2 = r^2 - h^2 = (8\ 00)^2 - (4\ 00)^2 = 48\ 00$ and $AE = \sqrt{48\ 00} = 6\ 93$

Furthermore, $AC = AE + EC = 6\ 93 + 6\ 93 = 13\ 86$ in

Thus, area $\triangle AOC = \frac{1}{2}(4)(13\ 86) = 27\ 72$ in ^{2}

Therefore, area segment $ABC = 66\ 97 - 27\ 72 = 39\ 25$ in ^{2}

A formula has been derived that provides an approximation of the area of a segment

$$A \cong \frac{4}{3}h_1^2\sqrt{\frac{2r}{h_1} - 0\ 61}$$

(The symbol $\cong$ is also used to indicate "approximately," to distinguish it from equal to) r is the radius of the circle and h_1 is the altitude of the segment EB

$$h_1 = OB - OE = r - h = 8\ 00 - 4\ 00 = 4\ 00 \text{ in}$$

Substituting

$$A \cong \frac{4}{3}(4\ 00)^2\sqrt{\frac{2(8\ 00)}{4\ 00} - 0\ 61}$$

$$= \frac{64\ 00}{3}\sqrt{4\ 00 - 0\ 61} = \frac{64\ 00}{3}\sqrt{3\ 39} = \frac{64 \times 1\ 84}{3} = 39\ 25 \text{ in}^2$$

The approximation yields results comparable to the regular solution

EXERCISES 15 4

1. Find the perimeter and area of a square with a diagonal equal to 8 00 in

2. Find the area of a trapezoid whose altitude is 10 00 in with bases of 6 00 in and 10 00 in , respectively

3. The length of an arc that measures 45° is 6 00 in Find the circumference of the circle and the area of the sector defined by the arc

4. Find the area of the segments formed by the sides of a regular hexagon and the circumscribing circle whose circumference is equal to 14π in. (Fig. 15-86.)

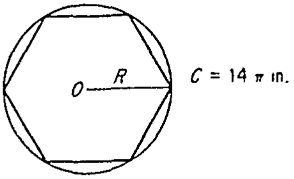


Figure 15-86

5. Which geometric figure will enclose the larger area, a regular octagon with an apothem $= 1\frac{3}{4}$ in.; a regular hexagon with a perimeter $= 12$ in., or a circle with a diameter $= 3\frac{5}{8}$ in.?

6. Which has a larger perimeter, a square whose area is 32 in.^{2} or a rhombus whose area is 32 in.^{2}? Justify your conclusion.

7. Find the area of a sector of a circle with a radius of 4.00 in. and an arc length equal to 4.00 in.

8. Find the central angle, interior angle, and sum of the interior angles for the following regular polygons. (a) heptagon, (b) nonagon, (c) decagon, and (d) dodecagon.

9. A circle has a diameter equal to 20.00 in. Find the area of the segment formed by an arc that measures 60° and its chord.

10. The perimeter of a regular octagon is 80.00 in. Find the circumference of both the inscribed and circumscribed circles defined by this octagon. Also, find the area of the ring formed by these two circles.

11. Given an isosceles trapezoid with bases of 12.00 in. and 18.00 in. and an area of 150.00 in.^{2}, find the altitude and length of the diagonals.

12. What is the relationship between the radius of the inscribed circle and circumscribed circle of an equilateral triangle?

13. Given the right triangle ABC ($\angle C = 90°$) with sides of 10.00 in., 8.00 in., and 6.00 in., find the area (Fig. 15-87).

14. Find the dimensions of the square, $ABCD$, if the area of $\triangle BOC =$ 16 in.^{2} (AC and DB are diagonals, Fig. 15-88.)

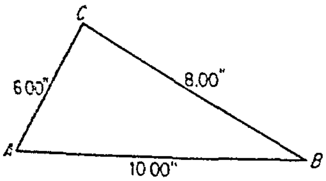


Figure 15-87

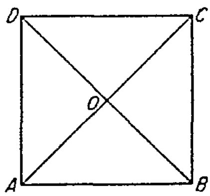


Figure 15-88

15. Identify the regular polygon defined by the following conditions:
   (a) central angle $= 30°$
   (b) interior angle $= 140°$

(c) sum of interior angles = 1,440°
(d) angle formed by apothem and radius of polygon = $22\frac{1}{2}°$
(e) diagonals are equal
(f) diagonals are perpendicular
(g) circumscribing circle has a diameter equal to the diagonal
(h) interior angle of one polygon is twice the interior angle of another

16. A circle with a diameter of 24 00 in circumscribes an equilateral triangle Find
(a) the length of the medians
(b) the radius of the inscribed circle

17. Two regular hexagons have sides of 6 00 in and 10 00 in, respectively Find
(a) the ratio of their apothems
(b) the ratio of their perimeters
(c) (compare these ratios to the ratio of the sides)
(d) the ratio of the areas
(e) (compare the ratio of part *d* to the ratio of the square of the apothems)

18 A regular polygon has an area of 60 00 in ^{2} Find the area of a similar polygon whose perimeter is two thirds of the perimeter of the first polygon

19. Construct, with compass and straight-edge (no other instruments needed) a regular octagon with an apothem equal to 5 units

20. What is the ratio of the perimeters of two regular heptagons if their radii are 7 00 in and 5 00 in, respectively

21. Given two regular octagons with the sides of one twice that of the other What is the ratio of their areas?

22. If one side of a regular octagon is 8 00 in and the side of another regular octagon is 20 00 in, what is the ratio of their radii and apothems?

23. Circles of radii 12 00 in and 18 00 in circumscribe two regular decagons, respectively Find the ratio of the apothems and the ratio of the areas of the decagons

24. Construct a regular octagon inscribed in a 1-in square

25. Construct a hexagon and a dodecagon with common circumscribing circles Use any convenient unit of measure

The following relationships apply to regular polygons
(a) Regular polygons with the same number of sides are similar
(b) The perimeter of similar polygons have the same ratio as any two corresponding sides

$$\frac{p_1}{p_2} = \frac{s_1}{s_2}$$

(c) Areas of regular polygons, with the same number of sides, are in the ratio of the squares of two corresponding sides

$$\frac{A_1}{A_2} = \frac{s_1^2}{s_2^2}$$

(d) The perimeters of regular polygons, with equal number of sides, have the same ratio as their radii or apothems.

$$\frac{p_1}{p_2} = \frac{R_1}{R_2} = \frac{r_1}{r_2}$$

15-8 SOLIDS: CYLINDERS-CONES-POLYHEDRONS

Polyhedrons are three-dimensional geometric forms whose faces or surfaces are polygons. These particular solids are classified as *prisms, pyramids,* and the *five regular polyhedrons.*

15-8a REGULAR POLYHEDRONS: *The five regular polyhedrons (polyhedra) are formed by surfaces that are regular (and congruent) polygons* (Fig. 15-89).

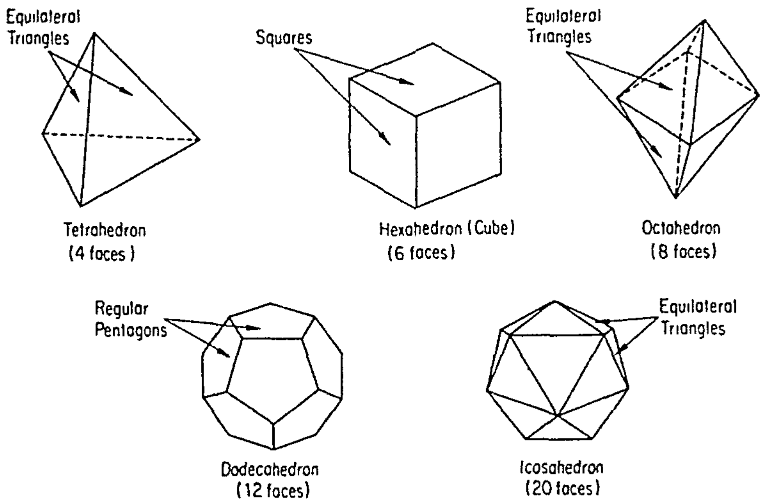
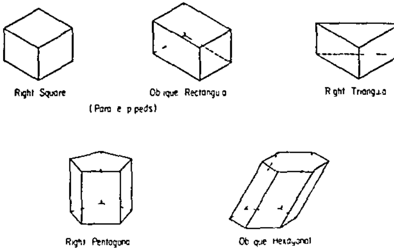


Figure 15-89

15-8b PRISMS: Prisms are *polyhedrons* with parallel congruent bases enclosed by parallelograms (surfaces). If the surfaces are perpendicular to the bases, the solids are called *right prisms.* Prisms with surfaces not perpendicular to the bases are classified as *oblique.* Prisms formed by bases constructed of parallelograms are referred to as *parallelepipeds.* If the bases are triangles, the prism is called a triangular prism; if the bases are hexagons, they are called hexagonal prisms, and so on (Fig. 15-90).

15-8c PYRAMIDS: *Pyramids* are polyhedrons with a base of three or more sides having the same number of triangular faces as there are sides in the base. The surfaces meet at a point called the *apex* (sometimes called the vertex) of the pyramid. A line from the apex to the center of the base is called the *axis* of the pyramid. If the axis is perpendicular to the base, it is also the altitude and this pyramid is classified as a *right pyramid.* A non-perpen-



PRISMS **Figure 15-90**

dicular axis defines an *oblique pyramid*. If the pyramid is cut off below the apex, it is said to be *truncated* and the portion containing the original base is referred to as a *frustrum*. A *regular pyramid* has a base that is a regular polygon and an altitude that is perpendicular at the center of the base (Fig. 15-91).

PYRAMIDS

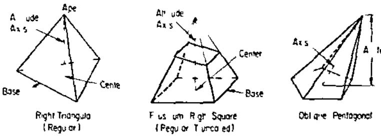


Figure 15-91

15-8d CYLINDERS AND CONES A *cylinder* is a geometric solid formed by a lateral surface wrapped around two parallel and equal circles called the bases. The *altitude* of a cylinder is the common perpendicular of the bases (Fig. 15-92).

CYLINDERS

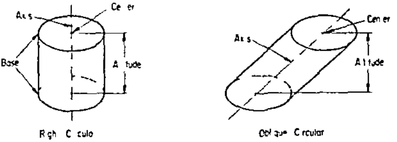


Figure 15-92

A *cone* is a geometric solid generated by rotating a line about a fixed point. In practice, the *upper nappe* or *lower nappe* are referred to as a *cone*. The *altitude* of a cone is a perpendicular from the vertex to the base (Fig. 15-93).

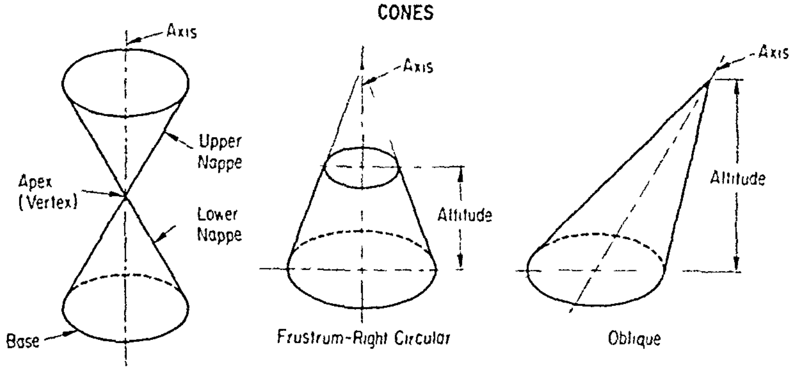


Figure 15-93

15-8e Miscellaneous Solids (Fig. 15-94):

Sphere: A *sphere* is a closed geometric surface whose points are equidistant from a fixed point called the center. The distances from the center to the surface are called radii.

Torus: A *torus* is a geometric solid formed by rotating a circle in a circular path around a fixed point.

Ellipsoid: An *ellipsoid* is a geometric solid formed by revolving an ellipse about its axis.

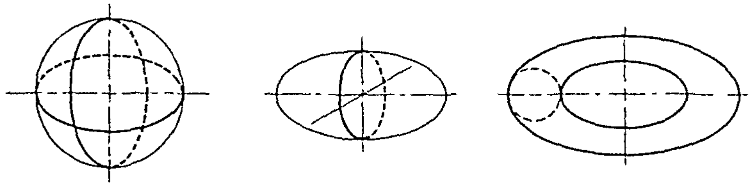


Figure 15-94 Sphere Ellipsoid Torus

15-8f Areas and Volumes of Solids:

A = area of base; L = area of lateral surface

T = total area $= L + 2A$; V = volume

$$A = \frac{\pi d^2}{4} = \pi r^2;\ L = \pi dh = 2\pi rh$$

$$T = \frac{2\pi d^2}{4} + \pi dh = \pi d\left(\frac{d}{2} + h\right) = 2\pi r(r + h)$$

$$V = \frac{\pi d^2 h}{4} = \pi r^2 h$$

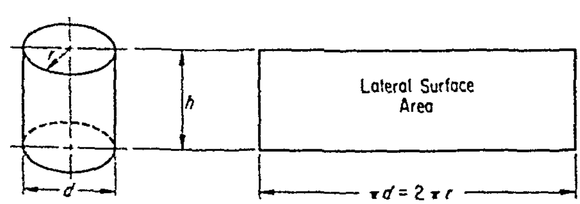


Figure 15-95

Cones-Pyramids

h = altitude; s = slant height; p = perimeter of base (The slant height of

a pyramid is the altitude of one of the faces; the slant height of a cone is the shortest distance from vertex to base, also referred to as an element of the cone.)

Cone:

$$L = \frac{1}{2}s(\pi d) = \pi rs$$

$$T = L + A = \frac{\pi ds}{2} + \frac{\pi d^2}{4} = \frac{\pi d}{4}(2s + d) = \pi r(s + r)$$

$$V = \frac{\pi d^2 h}{12} = \frac{\pi r^2 h}{3} = \frac{1}{3}(A)h$$

The volume of a circular cone is equal to one third the product of the area of the base and altitude

Pyramid:

$$L = \frac{1}{2}sp$$

$$T = L + A$$

$$V = \frac{1}{3}Ah$$

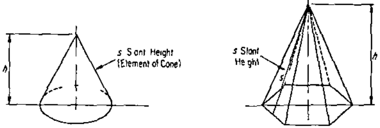


Figure 15-96

The volume of any pyramid is equal to one third the product of the area of the base and the altitude

The unit of measure for the volume of a geometric solid is the cube, cubic inches (in ^{3}), cubic centimeters (cm^3), and so on In determining the volume, units of measure of the various elements must be consistent in inches, feet, meters, and the like

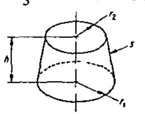
Frustrum of cone (Fig 15-97)

$$L = \pi s(r_1 + r_2)$$

$$T = L + A_1 + A_2 = \pi s(r_1 + r_2) + \pi r_1^2 + \pi r_2^2$$

A_1 = area of lower base, A_2 = area of upper base

$$V = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$$



r_1 = radius of lower base

r_2 = radius of upper base

s = slant height of frustrum

h = altitude

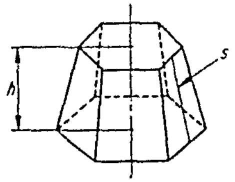
Figure 15-97

Frustrum of regular pyramid (Fig. 15-98)

$$L = \frac{1}{2}s(p_1 + p_2)$$

$$T = L + A_1 + A_2$$

$$V = \frac{1}{3}h(A_1 + A_2 + \sqrt{A_1 A_2})$$



P_1 = perimeter of lower base

P_2 = perimeter of upper base

h = altitude

s = slant height of frustrum

Figure 15-98

Prisms (Fig. 15-99)
Parallelepipeds (Fig. 15-100)

(m) $V = abc$

$T = 2ab + 2ac + 2bc = 2(ab + ac + bc)$

$d = \sqrt{a^2 + b^2 + c^2}$

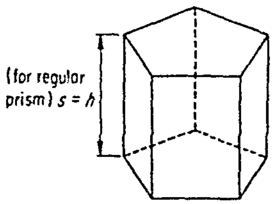
(n) If $a = b = c$, the parallelipiped becomes a cube and:

$V = a^3$

$T = 6a^2$

$d = a\sqrt{3}$

Sphere (Fig. 15-101)

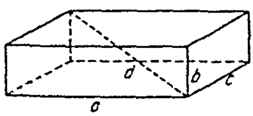


A = area base (cross-section)

$V = Ah$; $L = ps$; $T = 2A + L$

s = length of an edge

Figure 15-99



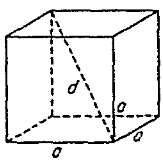
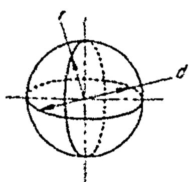
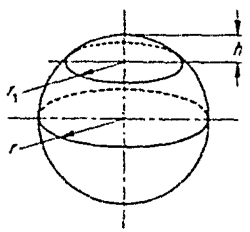


Figure 15-100





$V = \frac{4}{3}\pi r^3 = \frac{\pi}{6}d^3$

$T = 4\pi r^2 = \pi d^2$

Spherical Segment

$V = \frac{\pi h}{6}(3r_1^2 + h^2)$

$= \frac{\pi h^2}{3}(3r - h)$

Figure 15-101

EXAMPLE 15-N.

Find the lateral area, total surface area, and volume of a cylinder 36 00 in in height with a base radius of 12 00 in

Solution·

$$L = 2\pi rh = 2\pi(12\ 00 \times 36\ 00) = 2{,}712\ 96 \text{ in}^2$$
$$V = \pi r^2 h = \pi(12\ 00)^2(36\ 00) = 16{,}277\ 76 \text{ in}^3$$

Total surface area = lateral area + area of bases

$$= 2{,}712\ 96 + 2\pi(12\ 00)^2 = 3{,}617\ 28 \text{ in}^2$$

EXAMPLE 15-O

Find the volume and total surface area of a parallelipiped with a base 15 00 in by 30 00 in and an altitude equal to 36 00 in

Solution

$$V = abc = (15\ 00)(30\ 00)(36\ 00) = 16{,}200\ 00 \text{ in}^3$$
$$T = 2ab + 2ac + 2bc = 2(ab + ac + bc)$$
$$= 2(15\ 00 \times 30\ 00 + 15\ 00 \times 36\ 00 + 30\ 00 \times 36\ 00) = 4{,}140\ 00 \text{ in}^2$$

It might be of interest to study the physical properties of the containers of 15-*N* and 15-*O* (Fig 15-102)

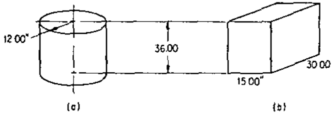


Figure 15-102

This indicates that the cylindrical container is much more economical to produce than the rectangular (parallelipiped) container—more volume with less material Actually, the container with maximum volume and minimum material is a right circular cylinder with an altitude (height) equal to the diameter of the base

EXAMPLE 15-P

Find the volume, lateral area, and total area of a regular hexagonal pyramid with an altitude of 10 00 in and sides of base equal to 6 00 in (Fig 15-103)

Solution·

The base of the pyramid is a regular hexagon whose area can be determined by the formula

$$A = \frac{3\sqrt{3}}{2}s^2 = 2.598s^2$$

$$A = 2.598(6.00)^2 = 93.53 \text{ in.}^2$$

$$V = \frac{1}{3}Ah = \frac{1}{3}(93.53)(10.00) = 311.76 \text{ in.}^3$$

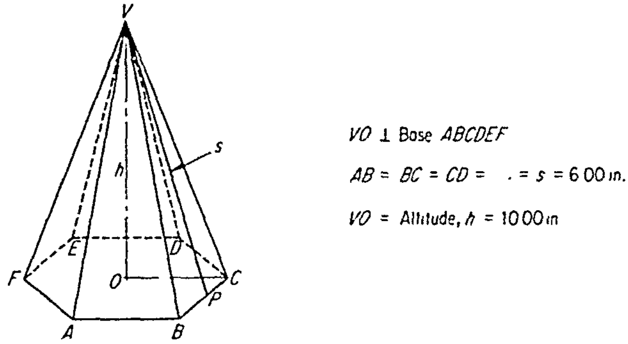
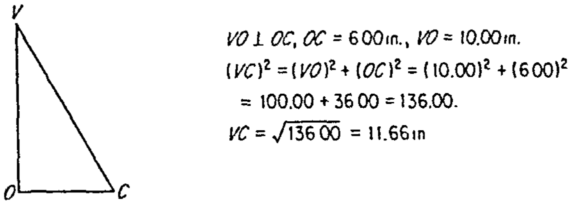


Figure 15-103a

In order to find the lateral area, the slant height must be determined initially. This involves a two-step procedure. First, the length of one of the edges, $VA = VB = VC, \ldots$, must be found. This can be accomplished by working with $\triangle VOC$, where the altitude, VO, is perpendicular to OC. Furthermore, OC is equal to the length of the side of the base, since the base is a regular hexagon; hence, $OC = 6.00$ in. (Fig. 15-103a.)



(a)

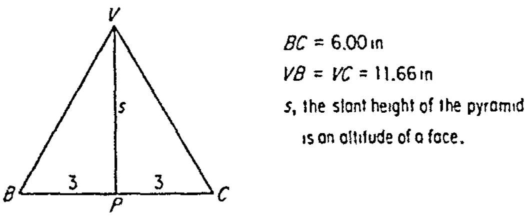


Figure 15-103b (b)

The second step involves finding the length of height, s. For this purpose, one of the faces, VBC, of the pyramid will be reconstructed (Fig. 15-103b). Furthermore, s is a $\perp$ bisector of BC, and $BP = PC = BC/2 = 3.00$ in. Thus,

$$s^2 = (VC)^2 - (PC)^2 = (11.66)^2 - (3.00)^2 = 136.00 - 9.00 = 127.00$$

and

$$s = \sqrt{127\ 00} = 11\ 27 \text{ in}$$

Therefore,

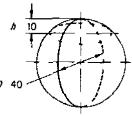
$$L = \frac{1}{2}sp = \frac{1}{2}(11\ 27)(36\ 00) = 202\ 86 \text{ in}^2$$

and

$$T = L + A_{\text{Base}} = 202\ 86 + 93\ 53 = 296\ 39 \text{ in}^2$$

EXAMPLE 15-Q

Find the capacity, in gallons, of a spherical tank 40 ft in diameter, along with the surface area Find, also, the volume of water when the level reaches a point 10 ft below the top (all dimensions are internal, Fig 15-104)



D - 40 ft r 20 ft
h = 10 f
1 ft³ 7 5 ga (approx)

Figure 15-104

Solution

$$V = \frac{\pi D^3}{6} = \frac{\pi(40)^3}{6} = 33{,}520 \text{ ft}^3$$

Capacity in gallons $= 33{,}520 \text{ ft}^3 \times 7\ 5 \text{ gal/ft}^3 = 251{,}140 \text{ gal}$
Volume of spherical segment

$$V = \frac{\pi h^2}{3}(3r - h) = \frac{\pi(10)^2}{3}(3 \times 20 - 10) = \frac{314}{3}(50) = 5{,}233 \text{ ft}^3$$

Volume of water at a level 10 ft below the top of tank

$$V = 33{,}520 - 5{,}233 = 28{,}287 \text{ ft}^3$$

Volume of water in gallons $= 28{,}287 \times 7\ 5 = 212{,}152$ gal
Total surface area $T = \pi d^2$

$$T = \pi(40)^2 = 5{,}024 \text{ ft}^2$$

EXERCISES 15 5

1. Water weighs approximately 62 4 lb/ft³ Find the weight of water in the tank in example 15-Q when the level is 5 ft above the center line

2. Find the volume, lateral area, and total surface area of two containers with the following specifications
   (a) Cylindrical height $= 7\frac{1}{4}$ in , diameter of base $= 6\frac{1}{2}$ in
   (b) Rectangular parallelepiped 4 in by $6\frac{1}{2}$ in by $9\frac{1}{4}$ in

These containers approximate a commercial gallon can. Compare the amount of tin (surface material) needed for each container.

3. Find the dimensions of a quart can with the diameter equal to the height (231 in.^{3} = 1 gal).

4. A right circular cone is inscribed in a right circular cylinder. Find the volume and total surface area of each if the diameter and height of the cylinder are equal to 10.00 in.

5. Compute the volume of a cylindrical water tank with a diameter of 20 ft and a height of 30 ft.

The force acting on the bottom of a container is given by the formula: $F = AhD$, where F is the force in pounds, A is the area of the bottom of the container, h is the height of the liquid, and D is the density of the liquid (62.4 lb/ft^3 for water). Find the force acting on the bottom of the tank when the tank is two thirds full.

6. The volume of a cube is equal to 64.00 cm^3. What is the volume of the inscribed sphere? Find also a diagonal of the cube.

7. Find the volume and total surface area of a regular pentagonal prism whose height is 14.0 in. and side of base is equal to 7.00 in.

8. A regular hexagonal prism with a height of 10 in., has a volume of 2,598 in.^{3} Find the lateral area.

9. Find the lateral area of a regular octagonal prism whose side of base is 9 cm and volume is equal to 1,448.4 cm^3.

10. A regular pyramid with a hexagonal base of side of 4.0 in. and an altitude equal to 6.0 in. is cut off (parallel to the base) 3.0 in. from the apex. Find the volume and total area of the frustrum and the pyramid that is cut off.

11. The slant height of a cone is 14.14 in. and the vertex angle is 90°. Find the volume and lateral area of this right circular cone.

12. A sphere is dropped into an open (full) cylinder of water and displaces 179.67 in.^{3} of water. The diameter of the sphere is equal to the height of the cylinder, and the total clearance between the walls of the cylinder and the surface of the sphere is 0.010 in. Find the diameter of the sphere and the internal dimensions of the cylinder.

13. A regular pentagonal pyramid is inscribed in a right circular cone with a slant height equal to 24.00 in. and base radius of 12.00 in. Find the volume and total area of the pyramid.

14. Find the weight of a gold bar that measures 8 cm by 14 cm by 30 cm (density of gold is 19.3 g/cm^3).

15. The volume of water at the indicated level of example 15-Q was taken when the temperature of the water was 60°F. Find the increase in volume for a 40°F rise in temperature. Coefficient of volume expansion for water can

be taken as 0 00012 increase in unit volume per degree Fahrenheit (Use either 1 ft^3 or 1 gal as the unit of volume)

REVIEW EXERCISES 15 6

1. Construct a line perpendicular to a line that bisects the coordinate axis Locate the new line 10 units away from the origin (4 solutions)

2. Construct two parallel lines 6 units apart Also, construct a transversal cutting these lines, such that one interior angle is twice the other interior angle

3. Construct an isosceles trapezoid with bases of 12 and 8 units, respectively, and sides equal to 6 units

4 Divide the trapezoid of exercise 3 into two trapezoids whose non parallel sides are in the ratio of 2 to 1 What is the length of the base of the smaller trapezoid?

5. Given $\underline{/\theta}$ (Fig 15 105), construct angles 2θ and 3θ

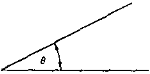


Figure 15-105

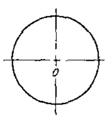


Figure 15-106

6. Construct a circle containing P_1 (3, 6) and P_2 (6, 3) and a line tangent to the circle and perpendicular to P_1P_2

7. Given circle O and point P, construct the tangents from P to circle O (Fig 15-106)

8 The volume of a right circular cone and a regular hexagonal pyramid is each 600 00 in^3 Find the dimension of the respective bases if the altitude of each is 10 00 in

9 The thickness of a spherical shell is 2 0 cm Find the volume of this shell having an outer diameter of 20 cm

10. Find the area of a regular octagon with side 10 00 in Find also the area of the segments formed by the sides and the circumscribing circle

11. Compare the volume and total area of two cylinders, one with a height twice its diameter and the other with a diameter twice its height (diameter remains constant)

12. The containers A, B, C are filled with liquids having densities of $D_a = 62\ 4\ lb/ft^3$, $D_b = 1\ 25\ g/cm^3$, and $D_c = 0\ 75\ g/cm^3$, respectively Find the weight of the corresponding liquids (Fig 15 107)

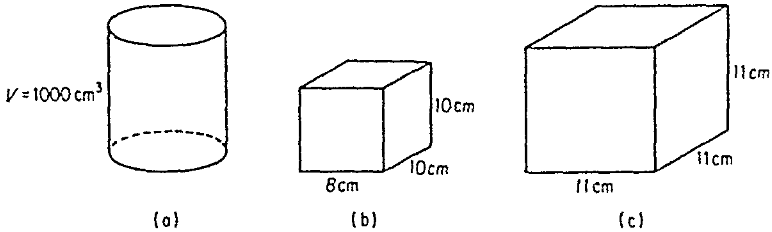


Figure 15-107

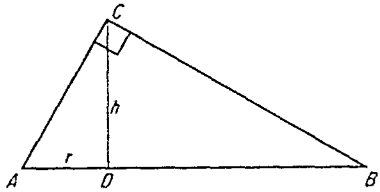


Figure 15-108

13. If $h = 8.0$ in. and $r = 6.0$ in., find AC, AB, BD, and the area $\triangle ABC$ (Fig. 15-108).

14. Construct an isosceles triangle whose altitude to the base is 7 units and whose base angles are 75°.

15. If $AB = 30$ ft, $AC = 270$ ft, and $BE = 10$ ft, find the height, h, of the monument (Fig. 15-109).

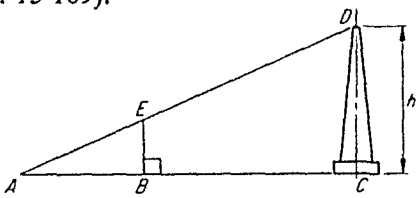


Figure 15-109

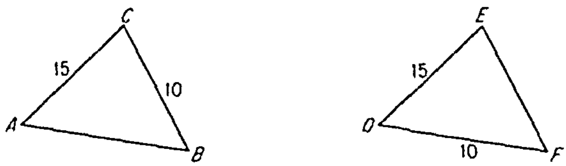


Figure 15-110

16. If $\triangle ABC \sim \triangle DFE$, find AB and FE (Fig. 15-110).

17. The edge of a regular quadrangular pyramid is equal to the length of a side of the base. Find the volume and total area if the side of the base is equal to 16.00 in.

18. Two spheres of diameters 8.0 in. and 12.0 in., respectively, touch each other while being supported on a plane. Find the distance, D, between the points of tangency (AB) of the spheres along the plane (Fig. 15-111).

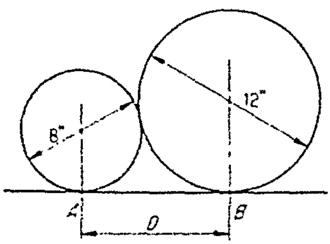


Figure 15-111

19. Find the volume of a regular tetrahedron whose slant height is equal to $10\sqrt{3}$ in

20. Find the surface area of a regular octahedron and a regular icosahedron whose edges measure 50 cm

21. Find the surface area of a regular dodecahedron if the apothem of a base is equal to 6 88 in

22. Which weighs more, a cylinder of lead with a diameter equal to its height, a gold sphere with a diameter equal to the diameter of the lead cylinder, or a cube of silver whose edge is equal in length to the diameter of the sphere (or cylinder)

Density of silver	10 5 g/cm³
Density of lead	11 34 g/cm³
Density of gold	19 3 g/cm³

23. Identify the geometric form (or forms) associated with the given relationships (23a is completed for illustrative purposes) L = lateral area, A = area, V = volume, p = perimeter, h = altitude, r = apothem or radius of circle, R = radius circumscribing circle, s = side, D = diameter, d = diagonal, b = base, l = length of arc, and S = sum of internal angles

(a) $d = \sqrt{2}s$ (diagonal of square)

(b) $A = \frac{1}{2}d_1d_2$

(c) $p = 8s$

(d) $A = \frac{\pi r^2\theta}{360^\circ}$

(e) $d = \sqrt{a^2 + b^2}$

(f) $p = a + b + c$

(g) $l = \frac{\pi r\theta}{180^\circ}$

(h) $\frac{a}{c} = \frac{h}{b}$

(i) $c = 2a$

(j) $r = l$

(k) $\angle A = \angle B = 45^\circ$

(l) $a = 3, b = 4, c = 5$

(m) $s = \frac{1}{2}\sqrt{d_1^2 + d_2^2}$

(n) $A = \frac{h(b_1 + b_2)}{2}$

(o) $\angle A = \angle B = \angle C$

(p) Central angle = 72°

(q) $A = \frac{1}{2}rp$

(r) $A = 3rs$

(s) $S = 1{,}080^\circ$

(t) $A = \frac{\pi D^2}{4}$

(u) $V = \pi r^2 h$

(v) $L = \frac{1}{2}sp$

(w) $L = \pi s(r_1 + r_2)$

(x) $V = \frac{1}{3}h(A_1 + A_2 + \sqrt{A_1A_2})$

(y) $c^2 = a^2 + b^2$

(z) $V = \frac{\pi D^3}{6}$

24. The lateral surface of a frustrum of a right circular cone, with a slant height of 10 00 in, is equal to 150π in² Find the volume if the radius of the lower base is twice the radius of the upper base

In exercises 25-27, use any convenient unit of measure

25. Construct two circles, tangent to each other, such that the circumference of one is twice the circumference of the other.

26. Construct two concentric circles such that the area of the first is twice the area of the second.

27. Construct two squares such that the area of the first is twice the area of the second.

28. Given a line 2 in. in length, construct a line $\sqrt{3}$ in. in length.

29. Sub-divide a plot of land, $ABCD$, into three parcels, all of equal areas. Inner boundaries are to be perpendicular to the bases (AB and CD). Give the dimensions of the parcels (Fig. 15-112).

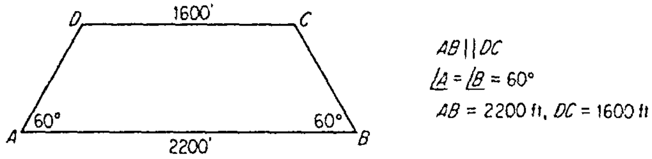


Figure 15-112

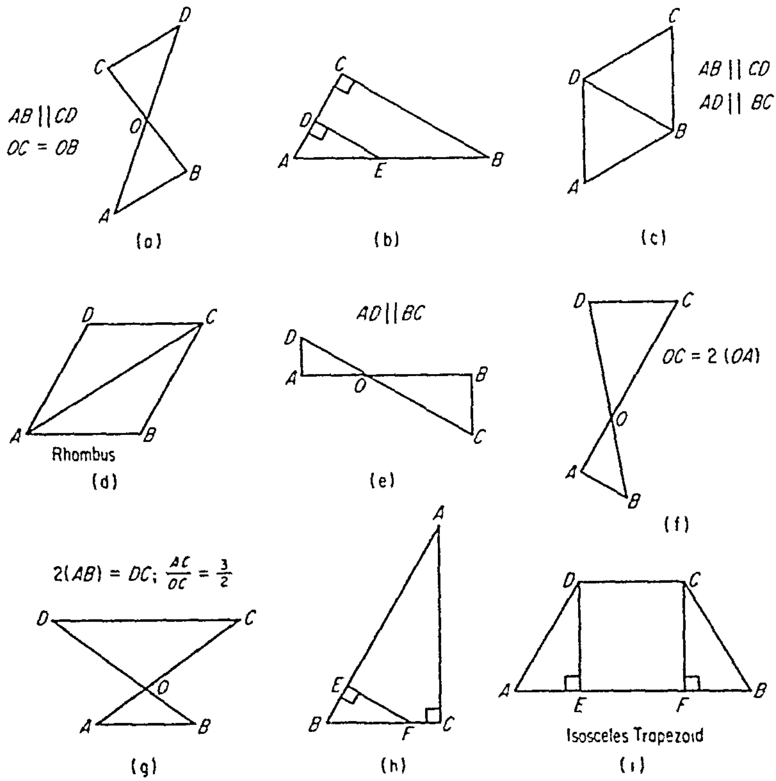


Figure 15-113

30. The geometric figures in Fig. 15-113 each contain a pair of triangles. Indicate the relationship of the triangles formed within the various forms as being either similar, congruent, or neither. Give reasons that may justify your conclusions.

Example Figure h $\triangle ABC \sim \triangle BFE$

Reason Both are right triangles with a common (equal) acute angle Thus, all angles are equal, a condition defining similar triangles

CHAPTER 16

Trigonometry

Trigonometry is a special branch of geometry (mathematics) that deals primarily with the relationship of the sides and angles of a triangle.

Fundamentally, the processes of trigonometry rely on the properties of similar triangles. Triangles are similar when the angles are equal and it follows that the corresponding sides are proportional. Furthermore, the ratio of any two corresponding sides of similar triangles is always constant, regardless of the length of the sides.

For example, the side opposite the 30° angle in a 30°-60°-90° triangle is one half the hypotenuse. Thus, the ratio of the side opposite the 30° angle to the hypotenuse is $\frac{1}{2}$; $CB/AB = \frac{10}{20} = \frac{1}{2}$, or $C'B'/A'B' = \frac{3}{6} = \frac{1}{2}$.

In the 45°-45° triangle the sides opposite the 45° angles are equal. Thus the ratio of the side opposite one of the 45° angles to the side adjacent to the same angle is equal to 1. $EF/DE = \frac{10}{10} = 1$, or $E'F'/D'E' = \frac{3}{3} = 1$ (Fig. 16-1).

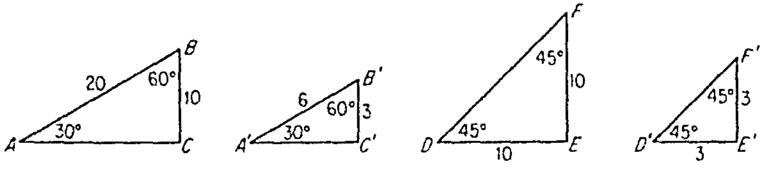


Figure 16-1

The use of trigonometric ratios is so extensive in the field of mathematics (and engineering) that they were translated into convenient definitions (names) designed to include all combinations (pairs) of sides. Furthermore, these definitions apply to all right triangles.

16-1 TRIGONOMETRIC FUNCTIONS

In a right triangle the ratio of the side opposite of an acute angle to the hypotenuse is called the *sine of the angle*, and abbreviated as sin. (pronounced sin, as in sign, and not sin, as in sinful.)

The ratio of the side adjacent an acute angle to the hypotenuse is called the *cosine of the angle*, and abbreviated as cos

The ratio of the side opposite an acute angle to the side adjacent to the same angle is called the *tangent of the angle*, and abbreviated as tan

The ratio of the side adjacent to the side opposite an acute angle is called the *cotangent*, abbreviated cot

The ratio of the hypotenuse to the side adjacent an acute angle is called the *secant*, or just sec

The ratio of the hypotenuse to the side opposite an acute angle is called the *cosecant*, abbreviated as csc (the abbreviations carry no period after them sin, cos, tan, cot, sec, and csc)

These are called the *six trigonometric functions* and will be further defined in terms of the right triangle ABC (Fig 16-2)

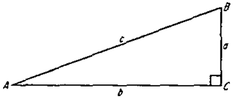


Figure 16-2

$$\text{sine of angle } A = \frac{\text{side opposite angle } A}{\text{hypotenuse}}$$

$$\sin A = \frac{a}{c}$$

$$\text{cosine of angle } A = \frac{\text{side adjacent angle } A}{\text{hypotenuse}}$$

$$\cos A = \frac{b}{c}$$

$$\text{tangent of angle } A = \frac{\text{side opposite angle } A}{\text{side adjacent angle } A}$$

$$\tan A = \frac{a}{b}$$

$$\text{cotangent of angle } A = \frac{\text{side adjacent angle } A}{\text{side opposite angle } A}$$

$$\cot A = \frac{b}{a}$$

$$\text{secant of angle } A = \frac{\text{hypotenuse}}{\text{side adjacent angle } A}$$

$$\sec A = \frac{c}{b}$$

$$\text{cosecant of angle } A = \frac{\text{hypotenuse}}{\text{side opposite angle } A}$$

$$\csc A = \frac{c}{a}$$

Likewise,

$$\sin B = \frac{b}{c},\ \cos B = \frac{a}{c},\ \tan B = \frac{b}{a},\ \csc B = \frac{c}{b},\ \sec B = \frac{c}{a},\ \cot B = \frac{a}{b}.$$

Notice that:

$$\sin A = \frac{a}{c} = \cos B$$

$$\tan A = \frac{a}{b} = \cot B$$

and

$$\sec A = \frac{c}{b} = \csc B$$

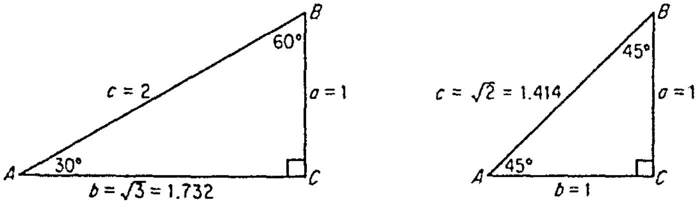
This states *that the function of an acute angle is equal to the co-function of its complement.* $\angle A + \angle B = 90°$, angle A is complementary to angle B, and vice versa. Thus, $\sin B = \cos A$, $\cot B = \tan A$, and so on, where sin and cos, tan and cot, and sec and csc are referred to as *co-functions*, respectively.

The values of the trigonometric functions for angles between 0° and 90° are given in Table I of the Appendix. With this table, as it applies to the trigonometric functions and the Pythagorean Theorem, all elements of a right triangle can be determined, if the size of an acute angle and the length of one side is known or the length of three sides are given.

The Pythagorean Theorem states *that in any right triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides*, or $c^2 = a^2 + b^2$.

The values of the trigonometric functions for the 30°-60° and 45° right triangles can be determined by applying definitions to known relationships involving the elements of the given triangles. These will be carried to three decimal places (Fig. 16-3).

Figure 16-3



$$\sin 30° = \frac{1}{2} = 0.500 \qquad \tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.577$$

$$\cos 30° = \frac{\sqrt{3}}{2} = 0.866 \qquad \cot 30° = \frac{\sqrt{3}}{1} = 1.732$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.155 \qquad \csc 60° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.155$$

$$\csc 30^\circ = \frac{2}{1} = 2\ 000$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = 0\ 866$$

$$\cos 60^\circ = \frac{1}{2} = 0\ 500$$

$$\tan 60^\circ = \frac{\sqrt{3}}{1} = 1\ 732$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0\ 577$$

$$\sec 60^\circ = \frac{2}{1} = 2\ 000$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0\ 707$$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = 0\ 707$$

$$\tan 45^\circ = \frac{1}{1} = 1\ 000$$

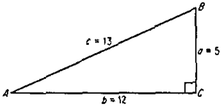
$$\cot 45^\circ = \frac{1}{1} = 1\ 000$$

$$\sec 45^\circ = \frac{\sqrt{2}}{1} = 1\ 414$$

$$\csc 45^\circ = \frac{\sqrt{2}}{1} = 1\ 414$$

EXAMPLE 16 A

If a right triangle has legs of 5 units and 12 units, respectively, find the six trigonometric functions (ratios) for both acute angles



Solution

A sketch representing the given data, will make the approach to the problem more meaningful Since two sides are given, the third can be determined by applying the Pythagorean Theorem

$$(AB)^2 = (AC)^2 + (BC)^2 = (12)^2 + (5)^2 = 144 + 25 = 169,$$

where $AB = 13$

Next, using the definitions and the numerical values associated with the respective sides will lead to the required ratios

$\sin A = \frac{5}{13}$	$\cos A = \frac{12}{13}$	$\tan A = \frac{5}{12}$	*Note*
$\csc A = \frac{13}{5}$	$\sec A = \frac{13}{12}$	$\cot A = \frac{12}{5}$	$\sin A = \frac{5}{13} = \cos B$
$\sin B = \frac{12}{13}$	$\cos B = \frac{5}{13}$	$\tan B = \frac{12}{5}$	$\tan A = \frac{5}{12} = \cot B$
$\csc B = \frac{13}{12}$	$\sec B = \frac{13}{5}$	$\cot B = \frac{5}{12}$	$\sec A = \frac{13}{12} = \csc B$

EXERCISES 16-1

Determine the indicated elements as listed, all of which are identified with Fig 16 4

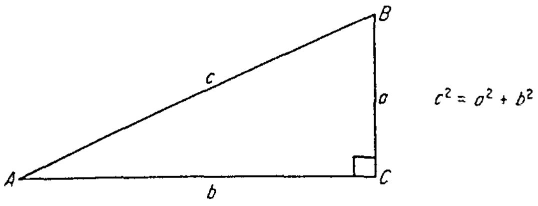


Figure 16-4

1. $a = 7$, $b = 24$, $c = 25$; find sin A, cos A, tan A
2. $a = 30$, $b = 40$; find c, sin A, cos B
3. $A = 30°$, $a = 5$; find sec B, csc A, cot B
4. $a = 7$, $b = 7$; find sin A, csc B, c
5. $a = 2$, $b = 4$; find cos A, tan B, cot A
6. $a = 4$, $b = 2$; find cos A, tan B, cot A
7. $a = 20$, $b = 15$; find sin A, cos B, tan A, cot B
8. $a = 17.32$, $b = 8.66$; find tan A, cot A, tan B
9. $a = 6\sqrt{3}$, $c = 12$; find sin A, sec B, csc A
10. $c = 42.42$, $a = 30.00$; find cos A, tan A, cot B

EXAMPLE 16-B:

Sin $A = 0.7000$. Find the other five trigonometric functions for this angle.

Solution:

Sketch the corresponding triangle defined by the given conditions. Since the sin of an angle is defined as the ratio of the side opposite an acute angle to the hypotenuse, any unit of measure can be used to construct this angle, if the ratio of the respective sides is equal to 0.700, whose fractional equivalent is, $\frac{7}{10}$, $\frac{70}{100}$, $\frac{35}{50}$, $\frac{14}{20}$.

Based on the principle of ratios, the sides can be measured in terms of inches, centimeters, feet, or a unit designed for convenience, such as indicated in Fig. 16-5.

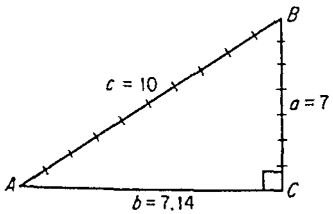


Figure 16-5

Next, find the remaining side (leg).

$$b^2 = (10.00)^2 - (7.00)^2 = 100.00 - 49.00 = 51.00$$
$$b = \sqrt{51.00} = 7.14$$

Applying the definitions of the trigonometric functions to the triangle under consideration, leads to completion of the problem.

$$\cos A = \frac{7\ 14}{10\ 00}, \qquad \tan A = \frac{7\ 00}{7\ 14}, \qquad \cot A = \frac{7\ 14}{7\ 00}$$

$$\sec A = \frac{10\ 00}{7\ 14}, \qquad \csc A = \frac{10\ 00}{7\ 00}$$

Usually, these ratios are reduced to decimal equivalents Presently, there is no need for this additional step

EXERCISES 16-2

Find the trigonometric properties as required (Fig 16-4)

1. $\cos A = 0\ 850$, find $\sin B$
2. $\tan A = 6\ 20$, find $\cot A$
3. $\cos A = 0\ 600$, find $\sin A$, $\tan A$, $\sec A$
4. $\tan A = 5\ 0$, find $\cot A$, $\cot B$, $\tan B$
5. $\sin A = \frac{3}{4}$, find $\sin B$, $\cos B$, $\tan B$
6. $\cot A = 0\ 777$, find $\tan A$, $\cot B$, $\tan B$
7. $\csc B = 2\ 000$, find $\sec A$, $\sin A$, $\cos B$
8. $\tan B = 1\frac{1}{4}$, find $\sin B$, $\cos B$, $\cot A$
9. $\sin A = \cos A$, find $\tan A$, $\cos B$, $\sin B$
10. $\tan B = 2\ 000$, find $\sin A$, $\cos A$, $\csc A$

16-2 TRIGONOMETRIC TABLES

The trigonometric functions associated with any angle can be determined from Table I in the Appendix These values have been developed by mathematicians and reprinted for use by the technician The tabulations of the natural functions are given for intervals of 1′ for all angles between 0° and 90° These ratios are carried out to five places In most fields of engineering, angular measurements within 5 are considered adequate

For studying purposes, a limited part of the table is reproduced in Fig 16-6 The table lists the angle and the numerical values of trigonometric functions Angles listed at the top of the page (0°-44°) refer to the respective trigonometric functions on the top of the page, whereas the angles on the bottom (45°-89°) refer to the functions listed at the bottom of the page The first column lists minutes ( ) with respect to angle and function at the top, whereas the extreme right column refers to the same properties at the bottom of the page

EXAMPLE 16 C

Find the sin 10°15′.

10°

′	Sin	Tan	Cot	Cos	′
0	.1737	.1763	5.6713	.9848	60
1	.1739	.1766	5.6617	.9848	59
2	.1742	.1769	5.6521	.9847	58
3	.1745	.1772	5.6425	.9847	57
4	.1748	.1775	5.6329	9846	56
5	.1751	.1778	5.6234	.9846	55
6	.1754	.1781	5.6140	.9845	54
7	.1757	.1784	5.6045	.9845	53
8	.1759	.1787	5.5951	.9844	52
9	.1762	.1790	5.5857	.9844	51
10	.1765	.1793	5.5764	.9843	50
11	.1768	.1796	5.5671	.9843	49
12	.1771	.1799	5.5578	.9842	48
13	.1774	.1802	5.5485	.9841	47
14	.1777	.1805	5.5393	.9841	46
15	$\boxed{.1779}$	.1808	5.5301	$\boxed{.9840}$	45
50	.1880	.1914	5.2257	.9822	10
51	.1882	.1917	5.2174	.9821	9
52	.1885	.1920	5.2092	.9821	8
53	.1888	.1923	5.2011	.9820	7
54	.1891	.1926	5.1929	.9820	6
55	.1894	.1929	5.1848	.9819	5
56	.1899	.1932	5.1767	.9819	4
57	.1900	.1935	5.1686	.9818	3
58	$\boxed{.1902}$	.1939	5.1606	.9817	2
59	.1905	.1941	5.1526	.9817	1
60	.1908	.1944	5.1446	.9816	0
′	Cos	Cot	Tan	Sin	′

Figure 16-6 79°

Solution:

The value of any trigonometric function is determined by cross-reference of angle versus function. Thus, sin 10°15′ is equal to 0.17794 and is usually written as sin 10°15′ = 0.17794. Regardless of dimensions, the ratio of the side opposite the 10°15′ angle to the hypotenuse remains constant at 0.17794 (Fig. 16-7).

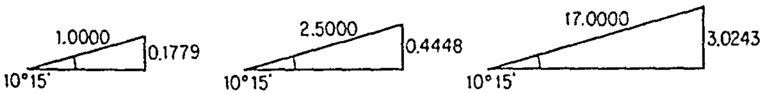


Figure 16-7

$$\sin 10°\ 15' = \frac{0.1779}{1.0000} = \frac{0.4448}{2.5000} = \frac{3.0243}{17.0000} = 0.1779 = (k)$$

EXAMPLE 16-D:

Find the cos 79°2′.

Solution:

The angle 79°2′ in terms of degrees is located at the bottom of the page, which means that the functions associated with this angle are listed at the bottom. Alignment of angle versus function gives a read out of 0.19024. Thus, cos 79°2′ = 0.19024.

Statements previously made concerning the ratio of the respective sides and the constant of proportionality, and so on, apply here as well.

Several other examples are

$$\tan 10°56' = 0\ 19317 \qquad \cot 79°4' = 0\ 19317$$

$$\cot 10° = 5\ 6713 \qquad \cos 10°60' = \cos 11° = 0\ 98163$$

The table of trigonometric functions can also be used to find an angle in terms of its function, such as $\cos \theta = 0\ 98404$ The problem now becomes one of finding an angle (θ) whose cos is 0 98404

EXAMPLE 16 E·

$\cos \theta = 0\ 98404$ Find θ

Solution

The first step is to locate in the column of cosines the value of 0 98404 The angle is determined on the basis of whether the function appears at the top of the page or at the bottom In this example, the cosine appears at the top Thus, $\theta = 10°15'$, or $\cos 10°15' = 0\ 98404$

Another way of writing the same statement is $\theta = \text{arc} \cos 0\ 98404$ This is read as θ is an angle whose cosine is 0 98404 The mathematical translation of "arc" is *an angle whose*

Frequently, a negative exponent is used to express the same statement Thus, $\theta = \cos^{-1} 0\ 98404$

Hence, the following three statements define the same concept

$$\cos \theta = 0\ 98404, \text{ where } \theta = 10°15'$$

$$\theta = \text{arc} \cos 0\ 98404, \text{ where } \theta = 10°15'$$

and

$$\theta = \cos^{-1} 0\ 98404, \text{ where, again, } \theta = 10°15'$$

Furthermore,

$$\sin \theta = 0\ 98218,\ \theta = 79°10$$

$$\tan \theta = 0\ 17783,\ \theta = 10°5$$

$$\cot \theta = 0\ 17783,\ \theta = 79°55$$

$$\theta = \text{arc} \sin 0\ 98179,\ \theta = 79°3'$$

$$\theta = \text{arc} \tan 5\ 2011,\ \theta = 79°7$$

$$\theta = \sin^{-1} 0\ 98163,\ \theta = 79°$$

$$\theta = \tan^{-1} 5\ 5671,\ \theta = 79°49'$$

$$\theta = \cot^{-1} 5\ 5671,\ \theta = 10°11'$$

$$\theta = \cos^{-1} 0\ 17365,\ \theta = 80°$$

$$\theta = \text{arc} \cot 5\ 1446,\ \theta = 11°$$

Using Table I of the Appendix, determine the value of the function or the angle as indicated in the following exercises. (Angles should be represented to the nearest minute.)

1. $\sin 30°10'$	*2.* $\cos 30°10'$
3. $\sin 59°52'$	*4.* $\cos 59°52'$
5. $\cot 89°55'$	*6.* $\tan 53°47'$
7. $\cos 36°35'$	*8.* $\sin 72°11'$
9. $\sec 45°$	*10.* $\csc 45°$
11. $\cos 0°$	*12.* $\sin 90°$
13. $\sin 0°$	*14.* $\cos 90°$
15. $\tan 47°15'$	*16.* $\tan \theta = 1.0818$
17. $\sin \theta = 1.0000$	*18.* $\sec \theta = 1.0000$
19. $\theta = \text{arc csc } 1.0000$	*20.* $\theta = \text{arc cot } 0.39223$
21. $\theta = \text{arc cos } 0.99924$	*22.* $\theta = \text{arc tan } 0.00320$
23. $\cot \theta = 0.67409$	*24.* $\theta = \tan^{-1} 1.8940$
25. $\theta = \sin^{-1} 0.90790$	*26.* $\theta = \cos^{-1} 0.01164$
27. $\cot \theta = 2.1123$	*28.* $\cos \theta = 0.46639$
29. $\csc \theta = 1.0000$	*30.* $\tan \theta = 0.0000$

16-3 SOLVING RIGHT TRIANGLES

Trigonometric functions evolved over a long period of mathematical history as suppositions turned into realities. One of the problems that stimulated the growth of the subject was the attempt to find the solution of the triangle. This meant finding unknown elements (angles-sides) of a triangle defined by limited information.

To define a triangle, at least three elements must be known, one of which must be the length of a side. Under these conditions, any triangle can be solved, although presently our concern will be directed or focused on the right triangle.

Again, to find the solution of a triangle, any one of these combinations is required to completely define the triangle:

(a) two sides and an angle
(b) two angles and a side; or
(c) three sides.

In engineering, the saying goes "If it can be constructed geometrically, it can be solved analytically."

This section is designed to provide the method and procedure of finding the solution of a right triangle Presently, if a triangle is other than a right triangle, it will have to be resolved by construction, into a right triangle

The following statements may serve as guidelines in solving a triangle

1 The largest angle is opposite the longest side and the smallest angle is opposite the shortest side
2 The sum of the acute angles of a right triangle is 90°
3 The hypotenuse squared is equal to the sum of the squares of the legs (This statement can be applied as a quick check to determine if, given three sides, the triangle is a right triangle)
4 Use an accurate drawing to check arithmetic computation, this approach may bring to light large descrepancies, if they exist
5 Use given data rather than information obtained through calculations whenever and wherever possible
6 Use values in tables as given, significant figures come into consideration on completing arithmetic operations involving given data and tabular data
7 Choose the function according to expediency of computation For example, if two sides are involved having dimensions of 1 175 in and 2 000 in , respectively, it is less trying to divide by 2 000 than by 1 175, thus, the function should be selected accordingly
8 Check results, not only graphically but also analytically, using relationships or data not used in the original computation
9 Round off angular measurements to the nearest 1

EXAMPLE 16 F

Given a right triangle with an acute angle equal to 32°10 and a hypotenuse equal to 12 000 in , determine the other elements (solve the triangle) (Fig 16-8)

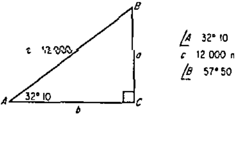


Figure 16-8

Solution

Make a fairly accurate sketch using given data Label the figure conveniently Next, determine what is required in terms of what is given Weigh the possible functions that involve the given parts and the unknown elements Proceed in steps, starting with an element that requires minimum involvement, such as $\angle B$

$$\angle A + \angle B = 90°, \text{ where } \angle A = 32°10$$

$$\angle B = 90° - \angle A = 89°60' - 32°10' = 57°50'$$

Side a can be found by using a function that involves an angle, the side opposite the angle, and the hypotenuse. There are two such functions, the sin and csc. Looking at both in terms of the given data provides these equations.

(a) $\sin 32°10' = \dfrac{a}{12.000}$, or $a = (12.000) \sin 32°10'$

(b) $\csc 32°10' = \dfrac{12.000}{a}$, or $a = \dfrac{12.000}{\csc 32°10'}$

It would appear that the first equation is simpler computationally than the second. For purposes of illustration, however, both calculations will be carried out.

$a = (12.000) \sin 32°10' = 12.000(0.53238) = 6.38856$ (rounded off) $= 6.389$

Using the csc leads to:

$$a = \frac{12.000}{1.8783} = 6.3887 = 6.389 \text{ in.}$$

To complete the problem, side b must be determined. This can be accomplished in several ways. For purposes of illustration, two procedures will be used: (a) the Pythagorean Theorem, and (b) the cos function.

(a) $c^2 = a^2 + b^2$, where $b = \sqrt{c^2 - a^2}$

$b = \sqrt{(12.000)^2 - (6.389)^2} = \sqrt{144.000 - 40.819} = \sqrt{103.181}$
$= 10.158$ in.

(b) $\cos 32°10' = \dfrac{b}{12.000}$, where $b = (12.000) \cos 32°10'$

Thus,

$$b = 12.000(0.84650) = 10.158 \text{ in.}$$

Both methods produce the same results.

The results can be checked by using elements that have not entered into the previous discussion. One approach would be to compare the ratio of the sides, 10.158 in. and 6.389 in., in terms of the value of tan 57°50′.

$$\tan B = \frac{10.158}{6.389} = 1.5899$$

From the tables, $\tan 57°50' = 1.5900$

This would indicate that the solution is acceptable.

EXAMPLE 16-G:

Given a triangle with angles of 42°20′ and 47°40′, respectively, with the side opposite the smaller angle equal to 16.000 in., find the remaining elements (solve the triangle).

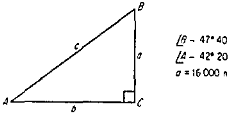


Figure 16-9

Solution.

$$\angle A + \angle B = 42°20' + 47°40' = 90°$$

Since $\angle A + \angle B = 90°$, $\angle C = 90°$ and $\triangle ACB$ is a right triangle

The hypotenuse c can be determined by using either the cos 47°40′ or sin 42°20′ in ratio with the known side a

$$\sin A = \frac{a}{c}, \text{ or } \sin 42°20' = \frac{16\,000}{c}$$

Furthermore,

$$c = \frac{16\,000}{\sin 42°20'} = \frac{16\,000}{0.67344} = 23.7584 = 23\,758 \text{ in}$$

The length of side b can be found by using the Pythagorean Theorem or a suitable trigonometric function In this example the latter approach will be applied

Again, there are four options with respect to use of a function and the given side

$$\tan A = \frac{a}{b}, \qquad \cot A = \frac{b}{a}$$

$$\tan B = \frac{b}{a}, \qquad \cot B = \frac{a}{b}$$

To minimize the rigors of arithmetic, the equation $\cot A = b/a$ will be used (same as $\tan B = b/a$)

$$\cot 42°20' = \frac{b}{16\,000}$$

or

$$b = 16\,000(1.0977) = 17.5632 = 17\,563 \text{ in}$$

The check will be left to the student

EXAMPLE 16-H:

If the sides of a right triangle measure, respectively, 50.00 in , 44.81 in , and 22.18 in , find the acute angles

Solution:

First make a sketch and then select a trigonometric function that will lead to defining an acute angle

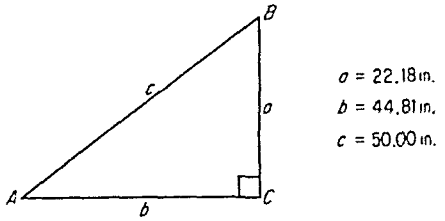


Figure 16-10

$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \sin B = \frac{b}{c}, \quad \text{or} \quad \cos B = \frac{a}{c}$$

The choice is arbitrary.

$$\sin A = \frac{a}{c} = \frac{22.18}{50.00} = 0.44360$$

From the tables of natural functions, arc sin 0.44360 = 26°20′.

It would appear that the remaining angle could be determined by simply subtracting 26°20′ from 90°. In practice, however, the second angle is determined independently of the first.

Thus,

$$\tan B = \frac{b}{a} = \frac{44.81}{22.18} = 2.02028 = 2.0203$$

and

$$\text{arc tan } 2.0203 = 63°40'$$

As a check: $\angle A + \angle B$ should be equal to 90°

$$26°20' + 63°40' = 89°60' = 90°$$

Measurements are approximations, approaching prescribed tolerances only. Mathematics is considered a perfect science; items of human construction are not. Computations involving technological concepts will be accurate within predetermined limits. These limits are usually based on the economics involved. Presently, the technician is expected to become conversant with the topic; industry will set the standards involving measurements.

EXERCISES 16-4

Solve the following right triangles, using Fig. 16-11 as reference. A sketch for each problem is advisable. Round off linear dimensions to correspond with given measurements. Angles should be expressed to the nearest minute.

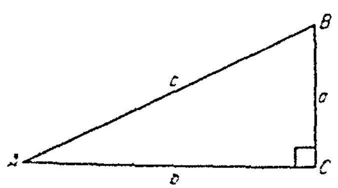


Figure 16-11

1. $c = 23\ 00$, $B = 60°00'$

2. $a = 7\ 07$, $c = 10\ 00$

3. $A = 42°05'$ $b = 11\ 07$ ft

4. $B = 65°15$ $c = 20\ 00$ in

5. $a = 35$ cm $b = 29\ 22$ cm

6. $b = 30\ 00$ m $c = 32\ 50$ m

7. $A = 79°$ $c = 10\ 120$ in

8. $B = 11°$ $c = 101\ 20$ cm

9. $A = 14°50'$ $a = 18\ 880$ in

10. $a = 4\ 02$ in $b = 4\ 02$ in

11. $c = 75$ m $a = 21$ m

12. $A = 86\ 4°$ $c = 76$ in

13. $c = 2$ $a = \frac{1}{2}$

14. $b = 1\frac{5}{8}$ $a = 2\frac{3}{8}$

15. $B = 36\ 7°$ $c = 120\ 00$

16-4 PRELIMINARY APPLICATION OF THE RIGHT TRIANGLE

Not all polygons are triangles, nor are all triangles right triangles If a polygon is defined (can be constructed), regardless of the limitations of available data or information, invariably all properties associated with the geometric figure can be determined One method by which this can be accomplished is by resolving the geometric form into a series of right triangles

Several problems that the technician may be faced with will be illustrated

EXAMPLE 16.1

Find the area of an isosceles triangle whose base is equal to 12 00 in and whose vertex angle is equal to 50°00

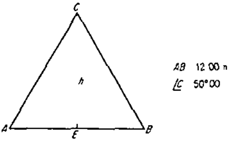


Figure 16-12

Solution:

A sketch is important, along with a review of the properties of an isosceles triangle, such as base angles are equal, $\angle A = \angle B$, and the altitude from the vertex to the base is the bisector of the vertex angle and the perpendicular bisector of the base: Thus,

$$\angle ECA = \angle BCE = \frac{\angle C}{2} = \frac{50°00'}{2} = 25°00'$$

Furthermore,

$$\angle A = \angle B = \frac{180° - \angle C}{2} = \frac{180° - 50°00'}{2} = \frac{130°00'}{2} = 65°00'$$

and

$$AE = EB = \frac{12.00 \text{ in.}}{2} = 6.00 \text{ in.}$$

Also,

$$\triangle AEC \cong \triangle BEC$$

and both are right triangles.

$$\text{Area } \triangle ABC = \frac{1}{2}h(AB)$$

The altitude, h, can be determined as an element of either congruent triangle.

$$\tan A = \frac{h}{AE} = \tan B = \frac{h}{EB}$$

Furthermore,

$$\tan 65°00' = \frac{h}{6.00}, \text{ or } h = 2.1445(6.00) = 12.8670 = 12.87 \text{ in.}$$

$$\text{Area } A = \frac{1}{2}(12.87)(6.00) = 38.61 \text{ in.}^2$$

EXAMPLE 16-J:

Surveyors are confronted, from time to time, with finding the height of an inaccessible object, using measurements taken on the ground or other surface. A typical example is shown in Fig. 16-13. The problem is to find the height of a steep hill from measurements taken at several points on a level surface below the hill. Whenever possible, convenient measurements are of primary concern.

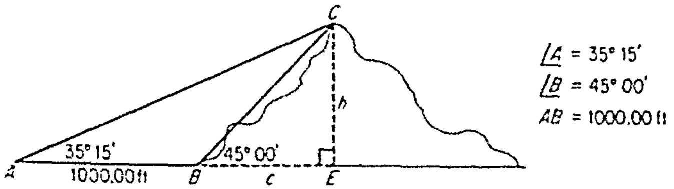


Figure 16-13

Solution

From point B on the ground, an angle is sighted to the top of the hill, point C One thousand feet away from B, in the same plane, another angle is sighted, this one from point A to point C The data is incorporated in the field sketch (Fig 16-13) To complete the figure, perpendicular CE or h is constructed to AB extended at E Let $CE = h$ and $BE = c$ It follows from $rt\triangle AEC$ and $rt\triangle BEC$ that

$$\tan 35°15 = \frac{h}{AE}, \text{ and } \tan 45°00 = \frac{h}{BE}, \text{ where } AE = 1{,}000\ 00 + c$$

$$h = (1{,}000\ 00 + c) \tan 35°15, \text{ and } h = (c) \tan 45°00$$

or

$$(1{,}000\ 00 + c) \tan 35°15 = (c) \tan 45°00$$

and

$$(1{,}000\ 00 + c)(0\ 70673) = 1\ 0000\ c$$

Furthermore

$$706\ 73 = 1\ 0000c - 0\ 70673c$$

$$706\ 73 = 0\ 2933c$$

Thus,

$$c = \frac{706\ 73}{0\ 2933} = 2{,}410\ 00 \text{ ft}$$

Since $\angle B = 45°$, it follows that $h = c = 2{,}410\ 00$ ft

By design, $\angle B$ was laid off conveniently at 45° This fact could have been utilized earlier in the computation

Recall

$$h = (1{,}000\ 00 + c) \tan 35°15$$

Therefore,

$$h = (1{,}000\ 00 + h) \tan 35°15 = (1{,}000\ 00 + h)(0\ 70673)$$

Thus,

$$h = 706\ 73 + 0\ 70673h$$

or

$$0\ 2933h = 706\ 73 \text{ and } h = \frac{706\ 73}{0\ 2933} = 2{,}410\ 00 \text{ ft}$$

EXAMPLE 16-K

From a point 15 00 in away from the center of a circle of radius 5 00 in, tangents are constructed Find the length of the tangents and the angle of tangency (Fig 16-14)

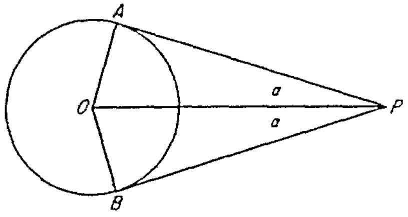


Figure 16-14

Solution:

$OA = 5{,}00$ in. $OP = 15.00$ in. AP and BP are tangent to the circle. Find $\angle BPA$ and PA or PB.

1. Sketch the geometric conditions.
2. Recall that a radius is perpendicular to a tangent at the point of tangency. Furthermore, a line from the center of the circle to the point from which the tangents are constructed will bisect the angle of tangency. Thus,

$$\angle OAP = 90°, \angle OBP = 90°$$

and

$$\angle OPA = \angle BPO = \alpha = \frac{\angle BPA}{2}$$

and

$$\triangle OAP \cong \triangle OBP$$

and both are right triangles. Also, $PA = PB$.

3. The length of tangent PA or PB can be determined by using the Pythagorean, Theorem, where $(OP)^2 = (PA)^2 + (OA)^2$, from which $(PA)^2 = (OP)^2 - (OA)^2$

$$(PA)^2 = (15.00)^2 - (5.00)^2 = 225.00 - 25.00 = 200.00$$
$$PA = \sqrt{200.00} = 14.14 \text{ in. } (PB = 14.14 \text{ in.})$$

4. The angle of tangency, $\angle BPA = 2\alpha$. α, can be computed by using the sin function.

$$\sin\alpha = \frac{OA}{OP} = \frac{5.00}{15.00} = 0.33333 \text{ and } \alpha = 19°28'$$

Therefore,

$$\angle BPA = 2\alpha = 2(19°28') = 38°56'$$

EXERCISES 16-5

Solve as indicated (round off angles to nearest minute).

1. $AB = 100.00$ in $\angle B = 55°$, $\angle A = 45°$ Find the area of $\triangle ABC$

2. Find h

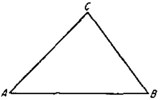


Figure 16 15

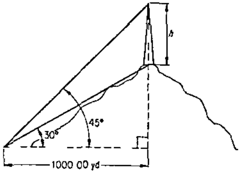


Figure 16-16

3. $AC = 16.00$ in , $\angle A = 29°50$ $\angle ABC = 129°50$ Find the area of $\triangle ABC$

4 $AC = 24.00$ in , $\angle CAB = 25°15$ $\angle ACB = 125°15'$ Find the area of $\triangle ABC$

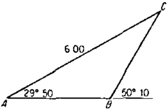


Figure 16-17

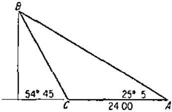


Figure 16-18

5. PA and PB are both tangent to circle O $\angle APB = 64°40$ The radius of circle $O = 3.00$ in Find the length of line segment OP

6. Find h

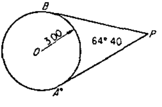


Figure 16-19

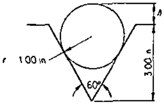


Figure 16-20

7. $BC = 320.00$ ft, $BD = 160.00$ ft $\angle B = 55°15$, $DE \parallel BC$ $\angle C = 90°$ Find AE and DE

8. $\triangle ABC$ is an equilateral triangle, $AB = 17.00$ in Find the radii of the inscribed and circumscribed circles

9. $AB = 16.00$ in , $BC = 8.00$ in , and $AC = 12.00$ in Find $\angle A$, $\angle B$, and $\angle C$

10. $D = 2.000$ in Find M and N

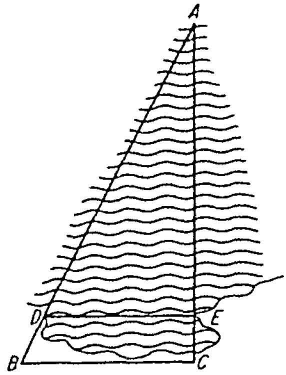


Figure 16-21

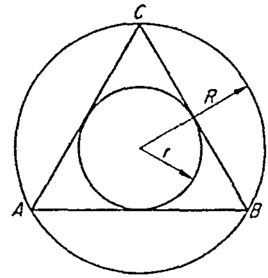


Figure 16-22

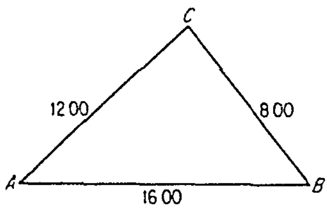


Figure 16-23

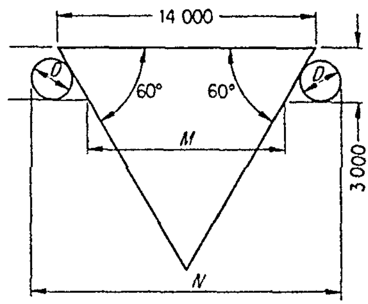


Figure 16-24

11. $ABCD$ is a parallelogram. $AB = 18.00$ in. and $BC = 9.00$ in. Find the area of $\square ABCD$ and d_1 and d_2.

12. $ABCD$ is a rhombus. $AB = 15.00$ in. and $\angle A = 45°$. Find the area of the rhombus and d_1 and d_2.

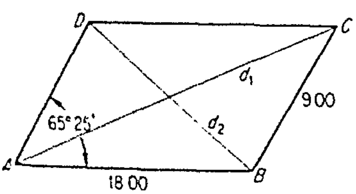


Figure 16-25

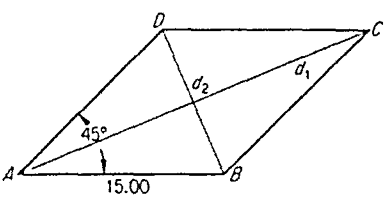
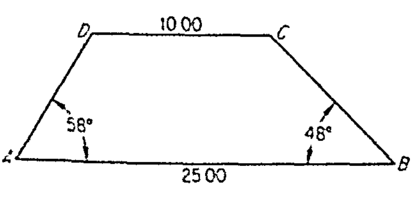


Figure 16-26

13. $ABCD$ is a trapezoid. $\angle A = 58°$ and $\angle B = 48°$. $AB = 25.00$ in. and $DC = 10.00$ in. Find the area of the trapezoid and the length of sides BC and AD.

Figure 16-27



14. Find the dimensions of A and B

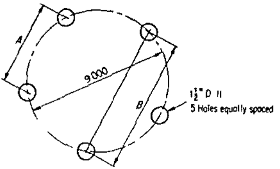


Figure 16-28

15. A survey was made on a small parcel of land, which resolves from field notes to Fig 16-29 Find the area of the plot, in acres, and the length of the northeast boundary, DC (1 acre = 43,560 ft^2) $AB = 1600\ 0$ ft and $BC = 1200\ 0$ ft $AD = 1000\ 0$ ft, $\angle A = 62°25'$, and $\angle B = 58°35$

16. A metal sphere, 5 00 cm in diameter, is dropped into a conical funnel whose elements form an angle of 56° The inside diameter of the funnel is equal to 8 50 cm (wide end) Will the top of the sphere (after it sets in the funnel) be above, below, or in line with the top of the funnel? If above or below, indicate by how much $D = 8\ 50$ cm, $d = 5\ 00$ cm, and the vertex angle = 56°

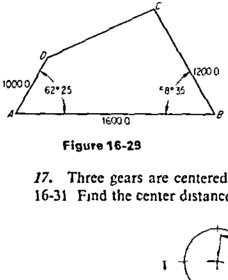


Figure 16-29

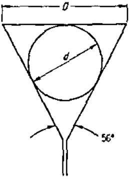


Figure 16-30

17. Three gears are centered according to the layout represented by Fig 16-31 Find the center distance A

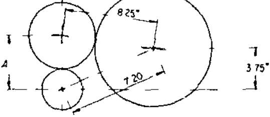


Figure 16 31

18. O is the center of a regular hexagon with apothem, $r = 6\ 00$ in Find

R, the radius of the hexagon; s, the side; A, the area of the hexagon; d, the diagonal; and P, the perimeter.

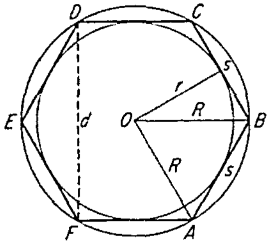


Figure 16-32

16-5 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

Many engineering concepts, such as those involving rotation, oscillation, vibration, and pulsation deal with angles usually larger than 90°. Regardless of the size of an angle, however, its function is still determined with respect to some corresponding acute angle. One way of analyzing the relationship of an angle that is more than 90° and its corresponding function in terms of an angle less than 90° is to represent the angles and functions graphically. The following illustration will establish a procedure whereby the various functions of large angles can be taken directly from the table of trigonometric functions.

To demonstrate the application suggested in the preceding paragraph, a circle with a radius equal to 1 unit will be constructed with its center at the origin of a pair of rectangular axes. Furthermore, the circumference of the circle will be divided into equal arcs of 15° and the axis into tenths of a unit, allowing for two-place accuracy only (Fig. 16-33).

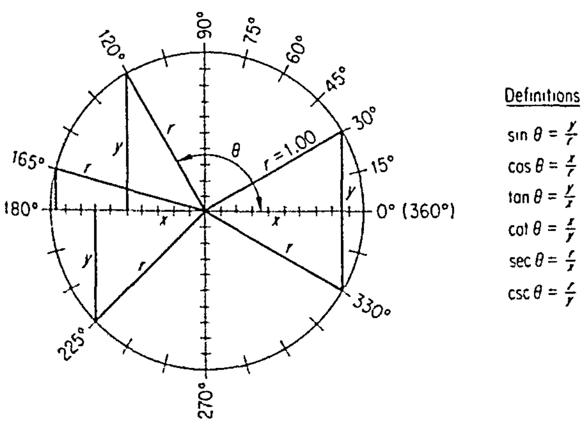


Figure 16-33

The figure can also be used to study the behavior of the six trigonometric functions as the angle (θ) increases from 0° to 90°.

Referring to the circle (Fig. 16-33) and applying definitions, the numerical values of the trigonometric functions for angles larger than 90° can be established accordingly.

$$\sin 120^\circ = \frac{y}{r} = \frac{0.87}{1.00} = 0.87$$

$$\cos 120^\circ = \frac{x}{r} = \frac{-0.50}{1.0} = -0.50$$

$$\tan 120^\circ = \frac{y}{x} = \frac{0.87}{-0.50} = -1.73$$

$$\cot 120^\circ = \frac{x}{y} = \frac{-0.50}{0.87} = -0.58$$

$$\sec 120^\circ = \frac{r}{x} = \frac{1.0}{-0.50} = -2.00$$

$$\csc 120^\circ = \frac{r}{y} = \frac{1.0}{0.87} = 1.15$$

It should be pointed out that graphical analysis provides only two place decimal approximations.

Note

$$\sin 60^\circ = 0.87,\ \cos 60^\circ = 0.50,\ \tan 60^\circ = 1.73$$

$$\csc 60^\circ = 1.15,\ \sec 60^\circ = 2.00,\ \cot 60^\circ = 0.58$$

Thus,

$$\sin 120^\circ = 0.87 = \sin 60^\circ$$

$$\cos 120^\circ = -0.50 = -\cos 60^\circ$$

$$\tan 120^\circ = -1.73 = -\tan 60^\circ$$

$$\cot 120^\circ = -0.58 = -\cot 60^\circ$$

$$\sec 120^\circ = -2.00 = -\sec 60^\circ$$

$$\csc 120^\circ = 1.15 = \csc 60^\circ$$

To further emphasize functional relationships of angles that fall into the second quadrant, $90^\circ < \theta \leq 180^\circ$, another relationship will be developed, using this time an angle equal to 165°.

$$\sin 165^\circ = \frac{y}{r} = \frac{0.26}{1.00} = 0.26$$

$$\cos 165^\circ = \frac{x}{r} = \frac{-0.97}{1.00} = -0.97$$

$$\tan 165^\circ = \frac{y}{x} = \frac{0.26}{-0.97} = -0.27$$

$$\cot 165^\circ = \frac{x}{y} = \frac{-0.97}{0.26} = -3.73$$

$$\sec 165^\circ = \frac{r}{x} = \frac{1.00}{-0.97} = -1.04$$

$$\csc 165^\circ = \frac{r}{y} = \frac{1.00}{0.26} = 3.86$$

Furthermore, the supplementary angle to 165° provides the following ratios.

$$\sin 15° = \frac{0.26}{1.00} = 0.26 \qquad \cos 15° = \frac{0.97}{1.00} = 0.97$$

$$\tan 15° = \frac{0.26}{0.97} = 0.27 \qquad \cot 15° = \frac{0.97}{0.26} = 3.73$$

$$\sec 15° = \frac{1.00}{0.97} = 1.04 \qquad \csc 15° = \frac{1.00}{0.26} = 3.86$$

Thus,

$$\sin 165° = \sin 15° \qquad \cos 165° = -\cos 15°$$

$$\tan 165° = -\tan 15° \qquad \cot 165° = -\cot 15°$$

$$\sec 165° = -\sec 15° \qquad \csc 165° = \csc 15°$$

The analysis just completed was intended to support the following generalization. For every angle θ, where $90° < \theta \leq 180°$, it follows that:

$$\begin{aligned}
\sin \theta &= \sin (180° - \theta) \\
\cos \theta &= -\cos (180° - \theta) \\
\tan \theta &= -\tan (180° - \theta) \\
\cot \theta &= -\cot (180° - \theta) \\
\sec \theta &= -\sec (180° - \theta) \\
\csc \theta &= \csc (180° - \theta)
\end{aligned}$$

Although these relationships can be stated as a rule, it is strongly suggested that the technician become familiar with the approach that leads to establishing these conditions. A quick sketch of the angle will not only help to find the value of the function in terms of an acute angle, but it will also be instrumental in determining the sign (+ or −) of the corresponding function.

For angles that fall into the third quadrant, where $180° < \theta \leq 270°$, the numerical values of the functions can be determined accordingly. Let $\theta = 225°$. It then follows that:

$$\sin 225° = \frac{-0.71}{1.00} = -0.71$$

$$\cos 225° = \frac{-0.71}{1.00} = -0.71$$

$$\tan 225° = \frac{-0.71}{-0.71} = 1.00$$

$$\cot 225° = \frac{-0.71}{-0.71} = 1.00$$

$$\sec 225^\circ = \frac{1.00}{-0.71} = -1.41$$

$$\csc 225^\circ = \frac{1.00}{-0.71} = -1.41$$

Previously it was established that (rounded to two places)

$$\sin 45^\circ = 0.7071 = 0.71,\ \cos 45^\circ = 0.71,\ \tan 45^\circ = 1.00$$
$$\cot 45^\circ = 1.00,\ \sec 45^\circ = 1.41,\ \csc 45^\circ = 1.41$$

Hence,

$$\sin 225^\circ = -0.71 = -\sin 45^\circ, \quad \cos 225^\circ = -0.71 = -\cos 45^\circ$$
$$\tan 225^\circ = 1.00 = \tan 45^\circ, \quad \cot 225^\circ = 1.00 = \cot 45^\circ$$
$$\sec 225^\circ = -1.41 = -\sec 45^\circ, \quad \csc 225^\circ = -1.41 = -\csc 45^\circ$$

Thus,

For every angle θ, where $180 < \theta \leq 270^\circ$

$$\sin \theta = -\sin (\theta - 180^\circ)$$
$$\cos \theta = -\cos (\theta - 180^\circ)$$
$$\tan \theta = \tan (\theta - 180^\circ)$$
$$\cot \theta = \cot (\theta - 180^\circ)$$
$$\sec \theta = -\sec (\theta - 180^\circ)$$
$$\csc \theta = -\csc (\theta - 180^\circ)$$

For angles that fall in the fourth quadrant, where $270 < \theta \leq 360^\circ$, the functional relationships are developed accordingly

Let $\theta = 330^\circ$ It then follows that

$$\sin 330^\circ = \frac{-0.50}{1.00} = -0.50 \qquad \cos 330^\circ = \frac{0.87}{1.00} = 0.87$$

$$\tan 330^\circ = \frac{-0.50}{0.87} = -0.58 \qquad \cot 330^\circ = \frac{0.87}{-0.50} = -1.73$$

$$\sec 330^\circ = \frac{1.00}{0.87} = 1.15 \qquad \csc 330^\circ = \frac{1.00}{-0.50} = -2.00$$

It has been previously established that (rounded to two decimal places)

$$\sin 30^\circ = 0.50, \quad \cos 30^\circ = 0.87, \quad \tan 30^\circ = 0.58$$
$$\csc 30^\circ = 2.00, \quad \sec 30^\circ = 1.15, \quad \cot 30^\circ = 1.73$$

Hence,

$$\sin 330^\circ = -0.50 = -\sin 30^\circ, \quad \cos 330^\circ = 0.87 = \cos 30^\circ$$

$$\tan 330° = -0.58 = -\tan 30°, \quad \cot 330° = -1.73 = -\cot 30°$$
$$\sec 330° = 1.16 = \sec 30°, \quad \csc 330° = -2.00 = -\csc 30°$$

Thus,

for every angle θ, where $270 < \theta \leq 360°$,

$$\sin\theta = -\sin(360° - \theta)$$
$$\cos\theta = \cos(360° - \theta)$$
$$\tan\theta = -\tan(360° - \theta)$$
$$\cot\theta = -\cot(360° - \theta)$$
$$\sec\theta = \sec(360° - \theta)$$
$$\csc\theta = -\csc(360° - \theta)$$

EXAMPLE 16-L:

Find the sin 138°, cos 138°, and tan 138°.

Solution:

A quick sketch indicates that the angle (138°) falls into the second quadrant, where the x coordinate (side adjacent) is negative.

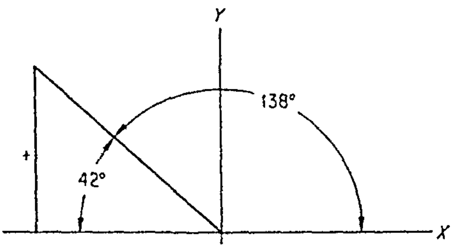


Figure 16-34

Thus,

$$\sin 138° = \sin(180° - 138°) = \sin 42° = 0.66913$$
$$\cos 138° = -\cos(180° - 138°) = -\cos 42° = -0.74314$$
$$\tan 138° = -\tan(180° - 138°) = -\tan 42° = -0.90040$$

EXAMPLE 16-M:

Find sin 351°15′, tan 351°15′, and cot 351°15′. (Fig. 16-35)

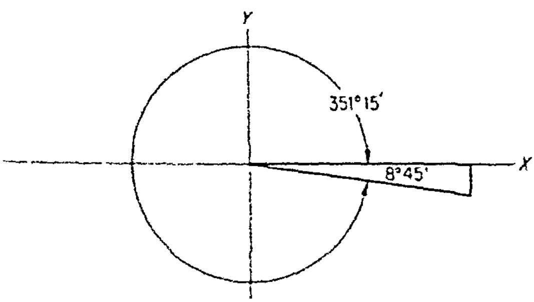


Figure 16-35

Solution·

$$\sin 351°15' = -\sin(359°60' - 351°15') = -\sin 8°45' = -0.15212$$

$$\tan 351°15' = -\tan(359°60' - 351°15') = -\tan 8°45' = -0.15391$$

$$\cot 351°15' = -\cot(359°60' - 351°15') = -\cot 8°45' = -6.4971$$

Angles formed by rotating the terminal side (radius) in a *counter-clockwise* direction are considered *positive* Angles formed by a *clockwise* rotation are considered *negative* The value of the function for negative angles can be determined by translating the negative angle to an equivalent positive angle This can be accomplished with reference to the position of the terminal side (Fig 16-36)

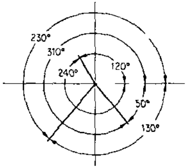


Figure 16-36

An angle of $+120°$ has the same terminal position as an angle of $-240°$ An angle of $-50°$ has the same terminal position as an angle of $+310°$, and this also holds for $-130°$ and $+230°$ Thus, $\sin(-240°) = \sin 120°$, $\cos(-50°) = \cos 310°$, and $\tan(-130°) = \tan 230°$,

EXAMPLE 16-N

Find $\sin(-225°)$, $\cos(-225°)$, and $\tan(-225°)$

Solution·

Figure 16-37 indicates that an angle of $-225°$ lies in the second quadrant and has the same terminal sides as an angle of 135° Therefore,

$$\sin(-225°) = \sin(135°) = \sin 45° = 0.70711$$

$$\cos(-225°) = \cos(135°) = -\cos 45° = -0.70711$$

$$\tan(-225°) = \tan(135°) = -\tan 45° = -1.0000$$

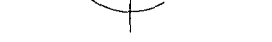


Figure 16-37

The following illustration is designed to study the behavior of the functions as the angle varies between 0° and 90°. The demonstration will deal only with the sine, cosine, and tangent; however, the study of the remaining functions can be completed by employing an analogous treatment.

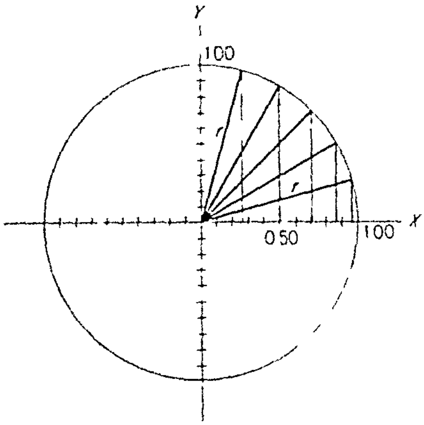


Figure 16-38

The radius r, called the radius vector, which is always considered positive, will be rotated from an initial position coincident with the horizontal axis to a terminal position coincident with the vertical axis.

As the radius vector generates the various angles, $0° \leq \theta \leq 90°$, the y coordinate increases from 0 to 1.00, whereas the x coordinate decreases from 1.00 to 0. However, the length of the radius, r, remains constant and equal to 1.

The table below lists variations in the elements along with corresponding values of the respective functions.

θ	x	y	r	$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$
0°	1.00	0.00	1.00	$\frac{0.00}{1.00} = 0.00$	$\frac{1.00}{1.00} = 1.00$	$\frac{0.00}{1.00} = 0.00$
15°	0.97	0.26	1.00	$\frac{0.26}{1.00} = 0.26$	$\frac{0.97}{1.00} = 0.97$	$\frac{0.26}{0.97} = 0.27$
30°	0.87	0.50	1.00	$\frac{0.50}{1.00} = 0.50$	$\frac{0.87}{1.00} = 0.87$	$\frac{0.50}{0.87} = 0.58$
45°	0.71	0.71	1.00	$\frac{0.71}{1.00} = 0.71$	$\frac{0.71}{1.00} = 0.71$	$\frac{0.71}{0.71} = 1.00$
60°	0.50	0.87	1.00	$\frac{0.87}{1.00} = 0.87$	$\frac{0.50}{1.00} = 0.50$	$\frac{0.87}{0.50} = 1.73$
75°	0.26	0.97	1.00	$\frac{0.97}{1.00} = 0.97$	$\frac{0.26}{1.00} = 0.26$	$\frac{0.97}{0.26} = 3.73$
85°	0.09	0.99	1.00	$\frac{0.99}{1.00} = 0.99$	$\frac{0.09}{1.00} = 0.09$	$\frac{0.99}{0.09} = 11.00$
90°	0.00	1.00	1.00	$\frac{1.00}{1.00} = 1.00$	$\frac{0.00}{1.00} = 0.00$	$\frac{1.00}{0.00} = \infty$

Figure 16-39

Conclusion:

As θ increases from 0° to 90°, $\sin\theta$ increases from 0 to 1.00. As θ increases from 0° to 90°, $\cos\theta$ decreases from 1.00 to 0.00. As θ increases from 0° to 90°, $\tan\theta$ increases from 0.00 to a value that cannot be determined. The symbol for a fraction with a denominator equal to zero is ∞, which is called *infinity*. Actually, the fraction (with denominator equal to zero) is not equal to infinity; it is just a convenient way of stating that the fraction

increases without limit The simplest way of expressing this symbol is to say that it is a number larger than any conceivable number Thus, the numerical value of the tangent increases without limit as the angle approaches 90°

$0° \leq \theta \leq 90°$	$0 \leq \sin\theta \leq 1$
$0° \leq \theta \leq 90°$	$1 \leq \cos\theta \leq 0$
$0° \leq \theta \leq 90°$	$0 \leq \tan\theta \leq \infty$

The study just completed was limited to the interval 0° — 90° However, the graphs of the various functions describe the behavior of the function for a complete cycle (Fig 16-40) Example 16 V illustrates the procedure for plotting trigonometric equations

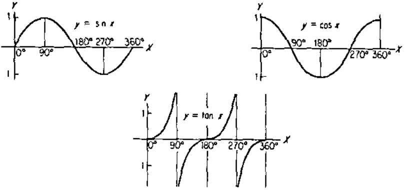


Figure 16-40

EXERCISES 16-6

Find the value of the functions for the various angles

1. sin 135°, cos 135°, tan 135°

2. cot 150°, sec 150°, csc 150°

3. sin 210°, cot 210°, sec 210°

4. cos 265°, tan 265°, cot 265°

5. sin 340°, cos 340°, cot 340°

6. cos 97°20′, tan 97°20′, sin 97°20′

7. sin 108°15′, cot 108°15′, cos 108°15′

8. tan 251°45′, cot 251°45′, cos 251°45′

9. sin 288°15′, cos 288°15′, cot 288°15′

10. sin (−97°20′), cot (−97°20), cos (−97°20′)

11. cos (−120°), tan (−120°), sec (−120°)

12. cot (−210°), cos (−210°), csc (−210°)

13. csc 137°35′, sin 137°35′, cos 137°35′

14. cos 172°40′, tan 172°40′, sin 172°40′

15. Determine the values for each trigonometric function for the quadrantal angles 180°, 270°, and 360°.

16. Find:
(a) sin (390°), cos (390°)
(b) tan (480°), cot (480°)
(c) sec (495°), csc (495°)
(d) sin (600°), cos (600°)

16-6 OBLIQUE TRIANGLES

Several convenient equations have been derived that lead to the solution of oblique triangles. *Oblique triangles* are those triangles that do not contain a right angle. Conditions involving the elements of an oblique triangle will be defined in reference to the general triangle (Fig. 16-41). The various formulas surrounding the general or oblique triangle apply also to the right triangle. It is more efficient and less trying, however, to use the direct method of section 16-3 when dealing strictly with the right triangle.

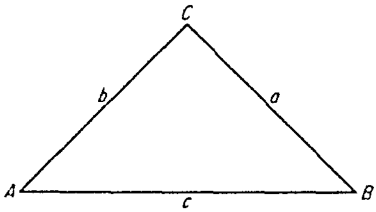


Figure 16-41

16-6a LAW OF SINES: This equation is very useful and applicable when two sides and an angle opposite one of the sides are known or two angles and a side are given. The Law of Sines can be stated as follows:

In any triangle the ratio of the sides to the sines of the angles opposite these sides is constant.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \text{ (for the given triangle)}$$

The Sine Law, as written above, can also be stated in terms of equivalent equations.

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \frac{a}{\sin A} = \frac{c}{\sin C}, \quad \frac{b}{\sin B} = \frac{c}{\sin C}$$

or the sides are proportional to the sines of the opposite angles.

EXAMPLE 16-O:

If $B = 110°$, $C = 40°$, and $c = 15.00$ in., find A and sides a and b.

Solution:

Sketch the triangle (Fig. 16-42) and label accordingly. Next, find $\angle A$.

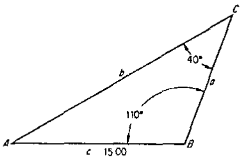


Figure 16 42

$$\angle A = 180^\circ - (\angle C + \angle B) = 180^\circ - (110^\circ + 40^\circ) = 30^\circ$$

Side b can be determined by applying the equation

$$\frac{b}{\sin B} = \frac{c}{\sin C}, \text{ where } b = c\frac{\sin B}{\sin C} = (15\ 00)\frac{\sin 110^\circ}{\sin 40^\circ}$$

$$\sin 110^\circ = \sin(180^\circ - 110^\circ) = \sin 70^\circ = 0\ 93969$$

$$\sin 40^\circ = 0\ 64279$$

Thus

$$b = \frac{15\ 00(0\ 93969)}{0\ 64279} = 21\ 90 \text{ in}$$

Furthermore, side a can be determined by using the relationship

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

where

$$a = c\frac{\sin A}{\sin C} = \frac{15\ 00 \sin 30^\circ}{\sin 40^\circ} = \frac{15\ 00(0\ 50000)}{0\ 64279} = 11\ 68 \text{ in}$$

As a check the corresponding ratios will be computed If these turn out to be constant, the solution of the triangle can be assumed correct

$$\frac{a}{\sin A} = \frac{11\ 68}{0\ 50000} = 23\ 36$$

$$\frac{b}{\sin B} = \frac{21\ 90}{0\ 93969} = 23\ 36$$

$$\frac{c}{\sin C} = \frac{15\ 00}{0\ 64279} = 23\ 36$$

Apparently the solution is correct

16-6b Law of Cosines (Parallelogram Law) The Law of Cosines is very useful when two sides and the included angle are given or when three sides of a triangle are known This law is sometimes regarded as an extension of the Pythagorean Theorem Furthermore, with slight modifica-

tion, the Law of Cosines is transferred into the Parallelogram Law, which deals with problems involving alternating current and the resolution of vectors. **Vectors** are symbols used to represent **forces**, velocities, and other physical properties having **magnitude** and **direction**. These symbols represent properties constructed to some scale corresponding to the unit of measurement of the respective technical elements.

For example, a force of 200 lb acting at an angle of 42° with a particular object is represented in Fig. 16-43, where 1 in. represents 100 lb.

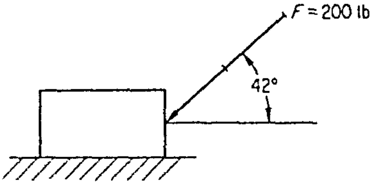


Figure 16-43

The Law of Cosines can be stated accordingly: *In any triangle, the square of any side is equal to the sum of the squares of the other two sides, minus twice the product of these two sides and the cosine of their included angle* (Fig. 16-41).

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Notice that when the included angle is 90°, $\cos 90° = 0$, and the Law of Cosines becomes the Pythagorean Theorem.

$$a^2 = b^2 + c^2 - 2bc\cos 90° = b^2 + c^2 - 2ac(0) = b^2 + c^2$$

When an exterior angle of a triangle is used instead of its corresponding interior angle, the Law of Cosines becomes what is known as **the Parallelogram Law**: $b^2 = a^2 + c^2 + 2ac\cos B'$ (Fig. 16-44).

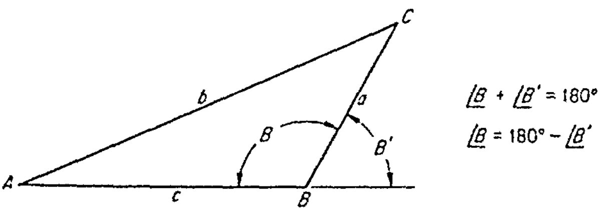


Figure 16-44

Notice the change in sign from minus to plus. This is justified on the basis that $\cos B = -\cos(180° - B) = -\cos B'$. Next, substituting this expression into the Law of Cosines, leads to the Parallelogram Law.

$$b^2 = a^2 + c^2 - 2ac\cos B = a^2 + c^2 - 2ac(-\cos B')$$
$$b^2 = a^2 + c^2 + 2ac\cos B' \text{ (Parallelogram Law)}$$

Similarly, it can be shown that

$$a^2 = b^2 + c^2 + 2bc \cos A$$
$$c^2 = a^2 + b^2 + 2ab \cos C$$

Another way of distinguishing between the Law of Cosines and the Parallelogram Law is in terms of the diagonals (d_1 and d_2) of a parallelogram (Fig 16 45)

$ABCD$ is a parallelogram
d_1 represents diagonal BD
d_2 represents diagonal AC
AB DC a AD BC b
$\angle A$ 180° $\angle B$ $\angle B > 90°$

Figure 16 45

$$(DB)^2 = d_1^2 = (AB)^2 + (AD)^2 - 2(AB)(AD)\cos A = a^2 + b^2 - 2ab\cos A$$
$$(AC)^2 = d_2^2 = (AB)^2 + (BC)^2 - 2(AB)(BC)\cos B = a^2 + b^2 - 2ab\cos B$$

But $\cos B = -\cos(180° - B) = -\cos A$ Thus, $d_2^2 = a^2 + b^2 + 2ab\cos A$

EXAMPLE 16 P

In parallelogram $ABCD$ in Fig 16 45 side $a = 12\ 00$ in side $b = 8\ 00$ in, and $\angle A = 50°$ Find the length of diagonals d_1 and d_2

Solution

From the law of cosines

$$d_1^2 = a^2 + b^2 - 2ab\cos A = (12\ 00)^2 + (8\ 00)^2 - 2(12\ 00)(8\ 00)\cos 50°$$
$$= 144\ 00 + 64\ 00 - 192\ 00(0\ 64279) = 84\ 58$$
$$d_1 = \sqrt{84\ 58} = 9\ 2 \text{ in}$$

(Notice that d_1 is also side BD of triangle ABD)
From the Parallelogram Law to the completion of the problem

$$d_2^2 = a^2 + b^2 + 2ab\cos 50°$$
$$= 144\ 00 + 64\ 00 + 192\ 00(0\ 64279) = 331\ 42$$
$$d_2 = \sqrt{331\ 42} = 18\ 2 \text{ in}$$

(Notice that d_2 can be viewed as the side AC of the triangle ABC)

EXAMPLE 16 Q

Figure 16 46 is a force diagram representing two concurrent forces acting on a pin connection of a truss Find the resultant force (magnitude and direction) and the stress developed in the pin The diameter of the pin

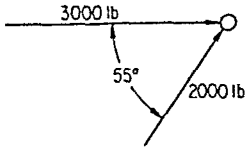


Figure 16-46

is equal to 1.00 in. The resultant of any two forces (in the same plane) acting on a given point can be represented by the diagonal of a parallelogram where the two vectors become the sides of the parallelogram.

Solution:

Construct a parallelogram with sides represented by the magnitude of the respective vectors and the vertex angle equal to the given angle (Fig. 16-47).

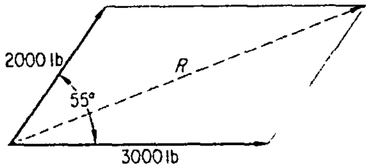


Figure 16-47

The resultant of this force system is represented by the diagonal R. Applying the Parallelogram Law leads to:

$$\begin{aligned} R^2 &= (3{,}000)^2 + (2{,}000)^2 + 2(3{,}000)(2{,}000)\cos 55^\circ \\ &= 9 \times 10^6 + 4 \times 10^6 + 12 \times 10^6(0.57358) \\ &= 13 \times 10^6 + 6.88 \times 10^6 = 19.88 \times 10^6 \\ R &= \sqrt{19.88 \times 10^6},\ R = 4{,}460 \text{ lb} \end{aligned}$$

The stress developed in the 1-in. pin can be computed by the formula $P = as$, where P is the force in pounds, a is the area in square inches and s is the stress in pounds per square inch.

$$s = \frac{P}{a} = \frac{4{,}460}{\frac{\pi(1)^2}{4}} = \frac{4{,}460 \times 4}{\pi} = 5{,}680 \text{ lb/in.}^2$$

The direction of the resultant force R is usually in reference to the angle it makes with the horizontal axis. (In this example, the 3,000-lb force is taken as the axis.) It can be computed by applying either the Law of Sines or the Law of Cosines; perhaps the Law of Sines seems more appropriate here. Figure 16-48 represents the elements of Fig. 16-46 in a manner more suited to the illustration of the Law of Sines.

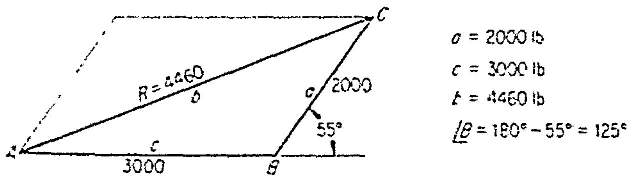


Figure 16-48

Thus,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

from which

$$\sin A = \frac{a}{b} \sin B$$

$$\sin B = \sin 125° = \sin (180° - 125°) = \sin 55°$$

$$\sin 55° = 0\ 81915$$

Therefore,

$$\sin A = \frac{2{,}000}{4{,}460}(0\ 81915) = 0\ 36733, \text{ and } A = 21°33'$$

Hence, the resultant of the given system is a 4,460 lb force acting at an angle of 21°33 with the horizontal axis

EXAMPLE 16 R

In structural design, whenever the loading is not extremely critical, graphical solutions of "force systems" are considered standard practice This method will be used to check the results of the preceding example

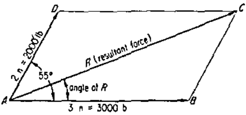


Figure 16 49

Solution

Step 1 Using a scale of 1 00 in $= 1{,}000$ lb, construct a horizontal line AB, where $AB = 3{,}000$ lb $= 3\ 00$ in
Step 2 With A as a vertex and AB as the initial side (using protractor) construct $\angle A = 55°$, with terminal side AD equal to 2,000 lb or 2 00 in
Step 3 Construct a parallelogram with sides DC and BC parallel and equal to AB and AD, respectively (forming parallelogram $ABCD$)
Step 4 Draw a diagonal from vertex A to the opposite vertex C
Step 5 Measure diagonal AC and $\angle BAC$, which corresponds to the magnitude and direction of the resultant, R (student should verify results)

EXERCISES 16-7

Solve the general triangle (find the missing elements) according to data given in exercises 1-6 (Fig 16-50) Express angles to nearest minute

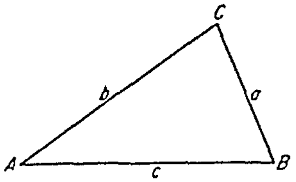


Figure 16-50

1. $A = 65°$, $B = 65°$, $a = 9.00$ in.
2. $C = 25°$, $B = 100°$, $b = 12.00$ cm
3. $a = 7.00$ in., $b = 10.00$ in., $C = 30°$
4. $a = 12.00$ in., $b = 13.00$ in., $c = 5.00$ in.
5. $A = 120°$, $a = 14.00$ in., $b = 11.00$ in.
6. $A = 42°15'$, $C = 59°45'$, $c = 8.50$ in.
7. Find the diagonals of a rhombus with sides equal to 20.00 in. and one interior angle equal to 45°.
8. For the parallelogram in Fig. 16-51, find diagonal BD and side DC.
9. Find the magnitude and direction of the resultant force of the force system represented in Fig. 16-52. Find, also, the stress in the rivet A created by the forces acting at that point. The diameter of the rivet is $\frac{3}{4}$ in. ($P = as$.)

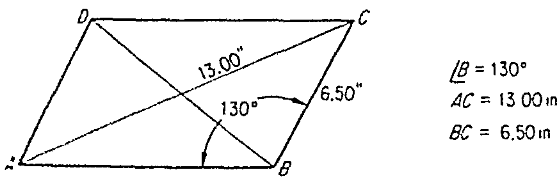


Figure 16-51

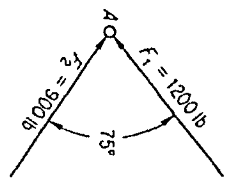


Figure 16-52

10. Determine graphically the magnitude and direction of the resultant force of the force system represented in Fig. 16-53.

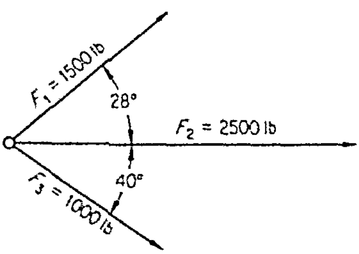


Figure 16-53

11. Two forces acting at right angles with each other produce a resultant force of 2,000 lb. One of the forces has a magnitude of 1,200 lb. Find the remaining force and the angle that the resultant makes with the unknown force.
12. Find $\angle R$ (Fig. 16-54). $RT = 42.00$ ft, $RS = 30.00$ ft.

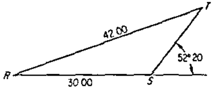


Figure 16-54

13. Find the distance between towers A and B, (Fig 16-55) $CA = 250$ rods, $\angle ACB = 67°15'$, and $CB = 375$ rods

14. A vector diagram for impedances in parallel is sketched in Fig 16-56 Find the total current of the branch (circuits) currents if $I_a = 3.10$ amps and $I_b = 4.80$ amps (Total current is equal to the vector sum of the branch currents)

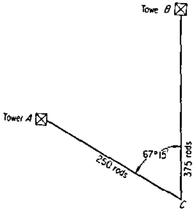


Figure 16-55

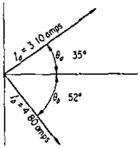


Figure 16-56

15. Determine graphically the resultant of the force system of Fig 16-57 (magnitude and direction)

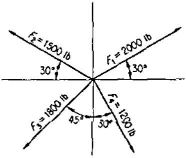


Figure 16-57

16-7 RADIAN MEASURE

Fundamentally, the angular unit of measure in dealing with arithmetic computations pertaining to geometric forms is the degree At times, it may be more convenient to use another form of angular measurement called the radian system of defining an angle *The radian (rdn) is defined as an angle subtended by an arc equal in length to the radius*

The circumference of a circle is equal to $2\pi r$ ($C = 2\pi r$), and in terms of

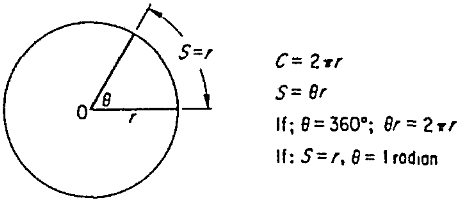


Figure 16-58

radians the length of arc S is equal to θr, ($S = \theta r$). It follows that when the central angle θ is equal to 360°, the length of the arc is equal to the circumference of the circle. This condition provides a basis for developing a factor of conversion whereby radian measures can be expressed in terms of angular degrees.

If $\theta = 360°$, the length of arc S is equal to the circumference C, or,

$$S = C$$

and

$$\theta r = 2\pi r$$

from which,

$$\theta = 2\pi$$

where θ is defined in radians.

Hence,

$$2\pi\ (\text{rdn}) = 360°$$

or

$$2\pi \text{ rdn} = 1 \text{ revolution}$$

Furthermore,

$$\pi \text{ rdn} = 180°$$

or

$$1 \text{ rdn} = \frac{180°}{\pi} = 57.2958° = 57°18'$$

It follows that

$$1° = \frac{\pi \text{ rdn}}{180} = 0.01745 \text{ rdn}$$

Thus, *to convert radians to equivalent degrees, multiply the expression in radians* by $180°/\pi$.

To convert degrees to equivalent radians, multiply the expression in degrees by $\pi/180°$.

Most often, when the radian measure is used, it appears in the form $\pi/2$, $3\pi/2$, $\pi/4$, . . . , rather than the decimal equivalent: 1.57, 4.71, 0.785, ... Whenever an angle measurement is indicated without units, it is understood to mean radians. Sin 2 would be interpreted as sin 2 rdn (sin 114.6° or sin 114°36′).

EXAMPLE 16 S

Express the following radian measures in terms of angular degrees $\pi/2$, $\pi/4$, $3\pi/2$, and $5\pi/4$

Solution

$1 \text{ rdn} = 180°/\pi$ is the equation for converting radians to degrees Thus, $\pi/2$ rdn are equivalent to

$$\frac{\pi}{2} \text{ rdn} = \frac{\pi}{2}\left(\frac{180°}{\pi}\right) = 90°$$

Similarly, $\pi/4$ rdn are equivalent to

$$\frac{\pi}{4} \text{ rdn} = \frac{\pi}{4}\left(\frac{180°}{\pi}\right) = 45°$$

Furthermore, $3\pi/2$ rdn are equal to

$$\frac{3\pi}{2}\left(\frac{180°}{\pi}\right) = 270°$$

and

$$\frac{5\pi}{4} \text{ rdn} = \frac{5\pi}{4}\left(\frac{180°}{\pi}\right) = 225°$$

EXAMPLE 16-T

Convert the following degree measurements into equivalent radian measurements $60°$, $90°$, $225°$

Solution

$1° = \pi/180$ rdn is the equation for converting degrees to radians Thus, $60°$ is equivalent to

$$60° = 60\left(\frac{\pi}{180}\right) = \frac{\pi}{3} \text{ rdn}$$

Similarly $90°$ is equivalent to

$$90° = 90\left(\frac{\pi}{180}\right) = \frac{\pi}{2}$$

Also,

$$225° = 225\left(\frac{\pi}{180}\right) = \frac{5\pi}{4} \text{ rdn}$$

EXAMPLE 16-U

Find the length of an arc subtended by an angle of $135°$ in a circle with a radius equal to 5 00 in

Solution:

The equation $S = \theta r$ is used to compute the length of an arc in terms of the central angle θ and the radius, where θ must be expressed in radians. The arc will be expressed in the same units as the radius.

Thus,

$$135^\circ = 135\left(\frac{\pi}{180}\right) = \frac{3\pi}{4} \text{ rdn}$$

and

$$S = \frac{3\pi}{4}(5.00) = \frac{15.00\pi}{4} = 11.78 \text{ in.}$$

(The radian, like π, carries no dimensional unit.)

The radian is the unit usually used in plotting the various trigonometric functions. The radian is also used in equations involving angular velocity, centrifugal force, and several electrical concepts.

EXAMPLE 16-V:

Plot the sine function $y = \sin x$ for the period 0° to 360°. The period, usually in radians, is plotted along the horizontal axis whereas the numerical values of the function are plotted along the vertical axis. For purposes of illustration, both the radian and degree will be used. Corresponding functional values will be taken from the proper tables.

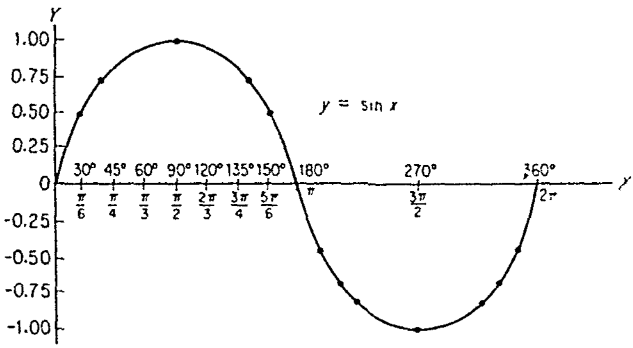


Figure 16-59

x (radians)	x (degrees)	$y = \sin x$
0	0°	0
$\frac{\pi}{6}$	30°	0.50
$\frac{\pi}{4}$	45°	0.71
$\frac{\pi}{3}$	60°	0.87
$\frac{\pi}{2}$	90°	1.00

x (radians)	x (degrees)	$y = \sin x$
$\frac{2\pi}{3}$	120°	0 87
$\frac{3\pi}{4}$	135°	0 71
$\frac{5\pi}{6}$	150°	0 50
π	180°	0 00
$\frac{7\pi}{6}$	210°	−0 50
$\frac{5\pi}{4}$	225°	−0 71
$\frac{4\pi}{3}$	240°	−0 87
$\frac{3\pi}{2}$	270°	−1 00
$\frac{5\pi}{3}$	300°	−0 87
$\frac{7\pi}{4}$	315°	−0 71
$\frac{11\pi}{6}$	330°	−0 50
2π	360°	0 00

EXAMPLE 16 W

Find the linear speed of a point on the rim of a flywheel (Fig 16 60) rotating at an angular velocity of 100 revolutions per minute (rpm) The diameter of the fly wheel is 16 00 in

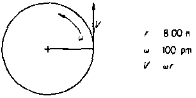


Figure 16 60

Solution

The linear velocity, v, of an object moving in a circular path is equal to the product of the angular velocity, ω, and the radius, r, of the circular path ($v = \omega r$), where v and ω are represented in the same units of time, and v and r are expressed in identical linear measurements Furthermore, ω is expressed in radians per unit of time

The angular velocity must be expressed in terms of radians per minute (or seconds)

$$1 \text{ revolution} = 2\pi \text{ rdn}$$

$$100 \text{ rpm} = 100(2\pi \text{ rdn}) = 200\pi \text{ rdn/min}$$

Thus,

$$v = 200\pi \text{ rdn/min} \times 8.00 \text{ in.} = 1600\pi \text{ in./min}$$

To convert inches per minute to feet per second, the following steps leading to proper conversion will be made:

$$12 \text{ in.} = 1 \text{ ft}$$

$$60 \text{ sec} = 1 \text{ min}$$

Hence,

$$v = 1{,}600\pi \text{ in./min} = 1{,}600\pi \frac{\text{in.}}{\text{min}}\left(\frac{1 \text{ ft}}{12 \text{ in.}}\right) = \frac{1{,}600\pi}{12} \text{ ft/min} = \frac{400\pi}{3} \text{ ft/min}$$

Furthermore,

$$v = \frac{400\pi}{3} \text{ ft/min} = \frac{400\pi}{3} \frac{\text{ft}}{\text{min}}\left(\frac{1 \text{ min}}{60 \text{ sec}}\right) = \frac{400\pi}{180} \text{ ft/sec}$$

$$= \frac{20\pi}{9} \text{ ft/sec} = 6.98 \text{ ft/sec}$$

Centrifugal Force

When an object is rotating along a circular path, it tends to pull away from the center of the path. The force with which the body acts away from the center is called **centrifugal force.** Several examples of centrifugal force are the tendency of an automobile (or railroad cars) to spin off the road when rounding a sharp curve at excessive speeds, the action of governors in regulating speeds of various engines, filtering liquids containing sediments by means of laboratory centrifuges, and, of course, the washing machine with its spin-dry cycle.

The equation for centrifugal force is:

$$CF = \frac{mv^2}{r}$$

where m is the mass, v the linear velocity, and r the radius of the path. In the English system, $m = W/g$, where W is the weight of the object in pounds and g is the pull of gravity, taken as 32.2 ft/sec^2.

EXAMPLE 16-X:

Find the tension in a thin wire, 24 in. long, fastened to a 2-lb sphere rotating in a circular path at 360 rpm. (Tension, T, in the wire is equal to the centrifugal force.)

Solution:

$$CF = \frac{mv^2}{r} = \left(\frac{W}{g}\right)\frac{v^2}{r}$$

$$r = 24 \text{ in.} = 2 \text{ ft}, \ W = 2 \text{ lb, and } g = 32.2 \text{ ft/sec}^2$$

Linear velocity, $v = \omega r$, where angular velocity is in terms of radians per second

Hence,

$$\omega = 2\pi n = \frac{2\pi 360}{60} = 12\pi \text{ rdn/sec}$$

Furthermore,

$$v = \omega r = 12\pi \text{ rdn/sec} \times 2 \text{ ft} = 24\pi \text{ ft/sec}$$

Therefore,

$$\text{Tension, } T = CF = \frac{2 \text{ lb}}{32\ 2 \text{ ft/sec}^2} \times \frac{(24\pi \text{ ft/sec})^2}{2 \text{ ft}}$$

$$= \frac{2(5{,}679\ 1)}{64\ 4} \times \frac{\text{lb} \times \text{ft}^2/\text{sec}^2}{\text{ft}^2/\text{sec}^2} = 176 \text{ lb}$$

EXERCISES 16-8

1. Express the following angles in terms of radians

(a) 30°	(f) 270°	(k) 75°
(b) 45°	(g) 360°	(l) 115°
(c) 60°	(h) 150°	(m) 200°
(d) 90°	(i) −60°	(n) −135°
(e) 180°	(j) 315°	(o) 450°

2. The following angles are defined in radians Convert to equivalent degree measure

a $\frac{\pi}{4}$	f $\frac{5\pi}{4}$	k 4 5
b $\frac{\pi}{3}$	g $\frac{5\pi}{3}$	l $\frac{4\pi}{3}$
c $\frac{\pi}{6}$	h $\frac{5\pi}{6}$	m $\frac{-\pi}{5}$
d $\frac{\pi}{2}$	i -3π	n $\frac{3\pi}{5}$
e $\frac{5\pi}{2}$	j $\frac{3\pi}{2}$	o 2

3. Plot the cosine curve, $y = \cos x$, for $0° \leq x \leq 360°$

4. Find the linear velocity in feet per second of a belt driven by a pulley with a diameter 6 00 in and an angular velocity of 240 rpm

5. An automobile is traveling at the rate of 60 miles/hr Find the angular velocity of the wheels if the tires have an outside diameter of 24 00 in

6. The linear velocity of a point 10 00 in away from the center of a flywheel is 120 ft/min Find the angular velocity (rpm) of the flywheel

7. Find the linear velocity (miles per hour) of a point on the equator. (Diameter of earth taken as 7,920 miles.)

8. Find the distance a point on the equator will travel during a time interval of 3 hr (rotation of earth).

9. Find the central angle of an arc with a length of 4.00 in. and a radius of 3.00 in.

10. How far will a point on a gear 7.50 in. away from its center travel as the gear makes one third of a revolution?

11. Find the length of the pendulum (Fig. 16-61) that swings through an arc of 10° and a distance (amplitude) of 6.00 in.

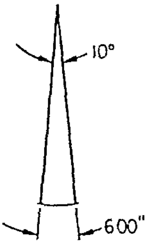


Figure 16-61

12. Find the force acting on a chord 3 ft long attached to an object weighing 5 lb, rotating in a circular path at a rate of 600 rpm.

16-8 TRIGONOMETRIC EQUATIONS AND IDENTITIES

An equation that is satisfied for every value of the variable is called an identity: $2(y+1) = 2y + 2$; $\sin\theta = 1/\csc\theta$; $\sec\theta = 1/\cos\theta$. Regardless of the value of the variable, the statement of equality holds true.

An equation that is satisfied by only certain values of the variable is called a conditional equation or simply an equation: $\sin\theta = \frac{1}{2}$, $3x + 1 = 7$, $\tan y = 1$. Trigonometric equations are conditional equations. The mathematical procedures established for algebraic expressions hold for trigonometric identities and equations.

16-8a TRIGONOMETRIC IDENTITIES Trigonometric identities are unconditional equations involving (usually) more than one trigonometric function and/or more than one angle. Identities are extremely useful mathematical expressions that are often used to reduce complex trigonometric relationships to more manageable statements.

There are eight fundamental trigonometric identities from which many other such statements can be derived. Only one of these identities will be verified, although the technician should accept the challenge and make an attempt to develop the others. This will help to analyze the concept and suggest procedures for expanding the list.

1. $\sin\theta = \dfrac{1}{\csc\theta}$ or $\csc\theta = \dfrac{1}{\sin\theta}$ and $\sin\theta\csc\theta = 1$

2 $\cos\theta = \dfrac{1}{\sec\theta}$ or $\sec\theta = \dfrac{1}{\cos\theta}$ and $\cos\theta\sec\theta = 1$

3 $\tan\theta = \dfrac{1}{\cot\theta}$ or $\cot\theta = \dfrac{1}{\tan\theta}$ and $\tan\theta\cot\theta = 1$

4 $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ or $\tan\theta\cos\theta = \sin\theta$

5 $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ or $\cot\theta\sin\theta = \cos\theta$

6 $\sin^2\theta + \cos^2\theta = 1$

7 $1 + \tan^2\theta = \sec^2\theta$

8 $1 + \cot^2\theta = \csc^2\theta$

9 $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

10 $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

EXAMPLE 16 Y

Prove or derive the identity $\sin^2\theta + \cos^2\theta = 1$

Solution

Using Fig 16-62 and definitions,

$$\sin\theta = \frac{y}{r} \quad \text{or} \quad y = r\sin\theta$$

$$\cos\theta = \frac{x}{r} \quad \text{or} \quad x = r\cos\theta$$

Applying the Pythagorean Theorem

$$x^2 + y^2 = r^2$$

but $y = r\sin\theta$ and $x = r\cos\theta$

Substituting the corresponding values of x and y, accordingly, leads to

$$x^2 + y^2 = r^2$$
$$(r\sin\theta)^2 + (r\cos\theta)^2 = r^2$$
$$r^2\sin^2\theta + r^2\cos^2\theta = r^2$$

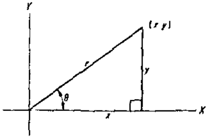


Figure 16-62

Dividing by r^2 leads to the identity $\sin^2 \theta + \cos^2 \theta = 1$.
The notation $\sin^2 \theta$ indicates that the value of the function associated with $\sin \theta$ is squared: $\sin^2 \theta = (\sin \theta)(\sin \theta)$. If $\theta = 30°$, $\sin \theta = \sin 30° = \frac{1}{2}$ and $\sin^2 30° = (\sin 30°)^2 = (\frac{1}{2})^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
Similarly, $\cos^3 \theta = (\cos \theta)(\cos \theta)(\cos \theta)$. If $\theta = 26°45'$, $\cos \theta = \cos 26°45' = 0.89298$ and $\cos^3 26°45' = (\cos 26°45')^3 = (0.89298)^3 = 0.71207$.

The following identities, 11-15, involve functions of double angles. It was indicated earlier that $\sin 2\theta \neq 2 \sin \theta$. With the additional identities, $\sin 2\theta$ and other such functions can be easily evaluated. More identities are listed in Table IV of the Appendix.

11. $\sin 2\theta = 2 \sin \theta \cos \theta$
12. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$
13. $\cos 2\theta = 1 - 2 \sin^2 \theta$
14. $\cos 2\theta = 2 \cos^2 \theta - 1$
15. $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}$

EXAMPLE 16-Z:

Find the $\tan 2\theta$, given $\theta = 40°$.

Solution:

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} \quad \text{(identity 15)}$$

$$\text{if } \theta = 40°, \tan 40° = 0.83910$$

Thus,

$$\tan 2\theta = \frac{2(0.83910)}{1 - (0.83910)^2} = \frac{1.67820}{0.29610} = 5.6713$$

This example was carried out to demonstrate procedures involving identities, rather than as a direct solution. Obviously, if $\theta = 40°$, then $2\theta = 80°$ and $\tan 2\theta = \tan 80°$. From the tables, it can be verified that $\tan 80° = 5.6713$.

EXAMPLE 16-AA:

Given the identity $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$, prove that $\cos 2\theta = 1 - 2 \sin^2 \theta$.

Solution:

In the process of proving or verifying an identity, suitable trigonometric substitutions can be made in accordance with the established principles of algebra.

Identity 6 states that $\sin^2 \theta + \cos^2 \theta = 1$, or $\cos^2 \theta = 1 - \sin^2 \theta$. Substituting this relationship ($\cos^2 \theta$) into the original statement leads to:

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \sin^2 \theta$$

Thus, the identity is verified

It should be pointed out that in the identities, θ is a variable and as such is a convenient way of expressing an element of a mathematical statement From time to time, as with other variables, it can assume other notations, such as x, y, α, 2θ, 30°, Once established, however, the same notation must be carried (with consistency) throughout the given discussion, where applicable

Hence, $\sin 3x = 1/\csc 3x$ and $\sin \theta = 1/\csc \theta$ are equivalent identities However, $\sin 3x = 1/\csc \theta$ is not an identity (assuming $3x \neq \theta$)

EXAMPLE 16 AB

Starting with the identity $\tan 2\theta = \sin 2\theta/\cos 2\theta$, prove that $\tan 2\theta = 2\tan\theta/(1-\tan^2\theta)$

Solution

From the list of identities, it appears that $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$ are appropriate for initial substitution

Hence,

$$\tan 2\theta = \frac{2\sin\theta\cos\theta}{1-2\sin^2\theta}$$

Next, divide both numerator and denominator of the expression on the right by $\cos^2\theta$

$$\tan 2\theta = \frac{\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}}{\dfrac{1-2\sin^2\theta}{\cos^2\theta}} = \frac{\dfrac{2\sin\theta\cos\theta}{\cos\theta\cos\theta}}{\dfrac{1}{\cos^2\theta} - \dfrac{2\sin^2\theta}{\cos^2\theta}}$$

but,

$$\frac{\sin\theta}{\cos\theta} = \tan\theta,\ \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta, \text{ and } \frac{1}{\cos^2\theta} = \sec^2\theta$$

Thus,

$$\tan 2\theta = \frac{2\tan\theta}{\sec^2\theta - 2\tan^2\theta}$$

Furthermore,

$$\sec^2\theta = 1 + \tan^2\theta$$

Therefore,

$$\tan 2\theta = \frac{2\tan\theta}{1+\tan^2\theta - 2\tan^2\theta} = \frac{2\tan\theta}{1-\tan^2\theta}$$

and the identity has been verified

16-8b TRIGONOMETRIC EQUATIONS A working knowledge of trigonometric identities can be considered one of the primary contributing factors in arriving at a solution of a trigonometric equation. It might be considered

an asset if the technician were able to memorize some of the identities. However, an education is not to be solely associated with the compartmentalized storage of related and unrelated facts but rather with the useful application of them. Instruments of memory can be found in the computer and the engineering handbook, with instant recall. The technician is expected to make productive analysis of information with meaningful projections. For example, a quick sketch and facility with the six trigonometric functions will provide the basis for developing those identities associated with the responsibilities of the technician. The point being: **analyze** not **memorize.**

Several suggestions will be offered that may be useful in bringing about a solution of trigonometric equations. Unfortunately, no procedure, to this point, has been developed that covers every conceivable situation—only a few generalizations.

1. Recognize that several angles will satisfy every equation. The workable interval must be predetermined, and only those values that fall within the given range are to be considered: $0 \leq \theta \leq 360°$; $-360° \leq \theta \leq 360°$, and so on.
2. Reduce the equation, if at all possible, to a single function and a single angle.
3. The processes of algebra are applicable to trigonometric equations.
4. If through application of various mathematical techniques the original equation is transformed into an equivalent equation, the **roots** are determined by **satisfying** the **original equation** rather than satisfying the derived equation alone. This, incidentally, is considered a part of the solution of trigonometric equations.
5. If a root of a derived equation fails to satisfy the conditions of the original equation, it is called an extraneous root.

EXAMPLE 16-AC:

Solve the equation $8 \sin^2 x = 2$ for the interval $0° \leq x \leq 720°$, or $0 \leq x \leq 4\pi$.

Solution:

The trigonometric equation $8 \sin^2 x = 2$ can be solved, initially, in the same manner as the algebraic equation $8x^2 = 2$, where

$$x^2 = \frac{2}{8} = \frac{1}{4} \quad \text{and} \quad x = \pm\frac{1}{2}$$

Thus,

$$8 \sin^2 x = 2$$

and

$$\sin^2 x = \frac{2}{8} = \frac{1}{4}$$

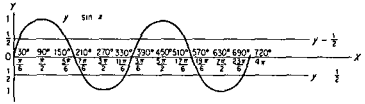
from which

$$\sin x = \pm\frac{1}{2}$$

Unlike the algebraic solution, which ends with $x = \pm\frac{1}{2}$, the roots of the trigonometric equation, $\sin x = \pm\frac{1}{2}$, depend on the interval under consideration Perhaps this can best be illustrated with reference to the graph $y = \sin x (0^\circ \leq x \leq 720^\circ)$

The intersection of the lines $y = \pm\frac{1}{2}$ and the curve $y = \sin x$ will define all the roots of $\sin x = \pm\frac{1}{2}$ for the interval $0^\circ \leq x \leq 720^\circ$

Thus, $x = \pi/6$, $5\pi/6$, $7\pi/6$, $11\pi/6$, $13\pi/6$, $17\pi/6$, $19\pi/6$, and $23\pi/6$ Or $x = 30^\circ$, 150°, 210°, 330°, 390°, 510°, 570°, and 690° These are the roots of the equation $8 \sin^2 x = 2$, where $0 \leq x \leq 4\pi$



EXAMPLE 16-AD:

Find the roots of the equation $\sin\theta = \cos\theta$ for the interval $0^\circ \leq \theta \leq 360^\circ$

Solution:

An equation involving several functions should be reduced to an equivalent equation in one function only This is usually accomplished by substituting various identities for certain terms of the original equation

Dividing the original equation by $\cos\theta$ leads to

$$\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\cos\theta} = 1$$

but

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

Thus,

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = 1$$

from which

$$\theta = 45^\circ \text{ and } 225^\circ$$

Or in terms of radians,

$$\theta = \frac{\pi}{4} \quad \text{and} \quad \frac{5\pi}{4}$$

The solution is incomplete until it is determined that the roots of the equivalent equation satisfy the original equation.

$$\sin\theta = \cos\theta, \text{ for } \theta = 45^\circ$$

$$\sin 45^\circ = 0.70711, \cos 45^\circ = 0.70711$$

Hence, $\theta = 45^\circ$ is a root.

Furthermore, $\sin 225^\circ = -0.70711$, $\cos 225^\circ = -0.70711$, and it appears that $\theta = 225^\circ$ is also a root.

Therefore, the roots of the equation $\sin\theta = \cos\theta$, for the interval $0^\circ \leq \theta \leq 360^\circ$, are $\theta = 45^\circ, 225^\circ$.

EXAMPLE 16-AE:

Solve the equation $\cos^2\theta - 2\sin\theta = 1$ for values of θ in the interval $0^\circ \leq \theta \leq 360^\circ$.

Solution:

An equivalent equation, in terms of one function, can be derived by taking advantage of the identity $\sin^2\theta + \cos^2\theta = 1$.

$$\sin^2\theta + \cos^2\theta = 1 \text{ and } \cos^2\theta = 1 - \sin^2\theta$$

Substituting $1 - \sin^2\theta$ for $\cos^2\theta$ in the original equation leads to:

$$\cos^2\theta - 2\sin\theta = 1$$

$$(1 - \sin^2\theta) - 2\sin\theta = 1$$

Simplifying:

$$1 - \sin^2\theta - 2\sin\theta = 1$$

$$-\sin^2\theta - 2\sin\theta = 0$$

and

$$\sin^2\theta + 2\sin\theta = 0$$

This expression can be solved by factoring and setting each factor equal to zero.

$$\sin\theta(\sin\theta + 2) = 0$$

$$\sin\theta = -2$$

and

$$\sin\theta = 0$$

Since there is no angle whose sine is -2, this is discarded (obviously it will not lead to a real root).

For $\sin\theta = 0$, $\theta = 0^\circ$, 180°, and 360°. Checking roots in the original equation:

$$\theta = 0^\circ$$

$$\cos^2\theta - 2\sin\theta = 1, \cos 0^\circ = 1, \sin 0^\circ = 0$$

$$(1)^2 - 2(0) = 1, \text{ and the equation is satisfied}$$

$$\theta = 180^\circ, \cos 180^\circ = -1, \sin 180^\circ = 0$$

Thus, $(-1)^2 - 2(0) = 1$, and this is also a root

$$\theta = 360^\circ, \cos 360^\circ = 1, \sin 360^\circ = 0$$

Hence, $(1)^2 - 2(0) = 1$, and the equation balances Therefore, the roots or solution of $\cos^2\theta - 2\sin\theta = 1$ are $\theta = 0^\circ$, 180°, and 360°

EXERCISES 16 9

Find the solutions of the following equation for the interval $0^\circ \leq \theta \leq 360^\circ$

1. $\sin\theta + \cos\theta = 0$

2. $\tan^2\theta = 1$

3. $\sin^2\theta = \frac{1}{4}$

4. $\cos^2\theta = \frac{1}{4}$

5. $3\cot\theta = 6$

6 $\sin\theta = \tan\theta$

7. $3\sin^2\theta + 2\sin\theta - 1 = 0$

8 $\cos\theta - \cos 2\theta = 1$

9. $4\sin^2\theta + 3\cos\theta = 4$

10. $\sec^2\theta - 2\tan\theta = 4$

11. $\sin 2\theta - 2\cos\theta = 0$

12. $\cos 2\theta - \sin^2\theta + 2 = 0$

13. $6\cos^2\theta - 7\cos\theta + 2 = 0$

14. $\sin\theta\cos\theta = \frac{1}{2}$

15. $\csc\theta\tan\theta = \frac{2}{3}$

Verify the following identities for the given angles (Ex 16-20)

16. $\sin 2\theta = 2\sin\theta\cos\theta$ $\qquad \theta = 75^\circ, 120^\circ$

17. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$ $\qquad \theta = 42^\circ, 225^\circ$

18. $\cos 2\theta = 2\cos^2\theta - 1$ $\qquad \theta = 0^\circ, \frac{\pi}{2}, \frac{3\pi}{2}$

19. $\tan^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{1 + \cos\theta}$ $\qquad \theta = 90^\circ, 135^\circ$

20. $\sin^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{2}$ $\qquad \theta = 0^\circ, 30^\circ, 120^\circ$

21. Given $\sin^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{2}$ and $\cos^2\dfrac{\theta}{2} = \dfrac{1 + \cos\theta}{2}$, prove $\tan^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

Verify the following identities.

22. $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$

23. $\sin\theta\cos\theta = \tan\theta\cos^2\theta$

24. $\sec^2\theta = \sin^2\theta + \tan^2\theta + \cos^2\theta$

25. $\sin\theta\cos\theta = \frac{\tan 2\theta\cos 2\theta}{2}$

REVIEW EXERCISES 16-10

1. In the isosceles triangle ABC (Fig. 16-63), $\angle C = 52°44'$ and $AB = 8.50$ in. Find the altitude, h, $\angle B$, and AC.

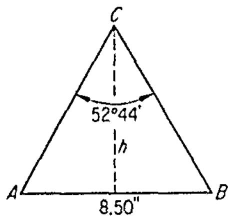


Figure 16-63

2. An isosceles triangle with an altitude equal to 11.00 in. has one angle equal to 72° and another equal to 36°. Find the length of the sides.

3. $ABCD$ (Fig. 16-64) is a parallelogram. $\angle A = 58°12'$ and $AC = 12.50$ in. Find AB and AD.

4. Find $\angle A$ and $\angle C$ (Fig. 16-65).

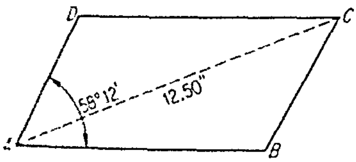


Figure 16-64

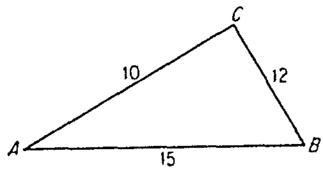


Figure 16-65

5. In the regular octagon in Fig. 16-66, $R = 16.00$ in. Find the radius, r, of the inscribed circle and the perimeter of the octagon.

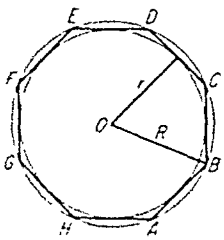


Figure 16-66

6. The radius of the inscribed circle of a regular pentagon is 12.00 in. Find the radius of the circumscribing circle and the perimeter of the pentagon.

7. Find the height of the tower, h, in Fig. 16-67.

8. Find the resultant of the force system, graphically (Fig. 16-68).

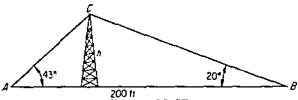


Figure 16-67

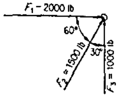


Figure 16-68

9. In Fig 16-69, $AC = 10\ 25$, $BC = 8\ 75$, and $AB = 14\ 00$ Find h (altitude) and AD

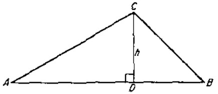


Figure 16-69

10. The three gears A, B, and C in Fig 16-70 are in mesh and their diameters are $D_a = 11\ 500$ in , $D_b = 7\ 000$ in , and $D_c = 14\ 000$ in Find dimensions X and Y

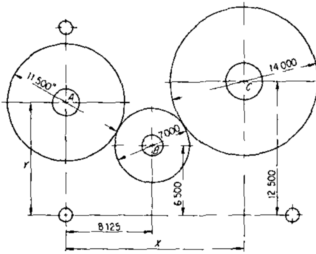


Figure 16-70

11. A metal strip is rolled out between two heavy rollers (Fig 16-71) If the sheet moves at the rate of 50 ft/min, what is the angular velocity (rpm) of each roller? Through what angle will the larger roller rotate as the smaller roller makes 1 revolution? (Neglect friction)

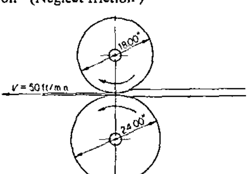


Figure 16-71

12. Find the velocity with which block A slides when the angular velocity of the oscillating arm is equal to 60 rpm and block A is in the position indicated in Fig. 16-72. Determine also the velocity of block A when it reaches the mid-point, M, of its horizontal motion.

13. In Fig. 16-73, $\angle BAD = 10°$, $\angle CAB = 20°$, and $AC = 18.2$ ft. Find BC.

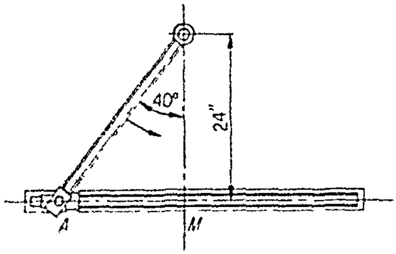


Figure 16-72

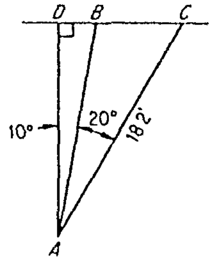


Figure 16-73

14. In circle O in Fig. 16-74, the inscribed $\angle ACB = 90°$, $AC = 10.00$ in., and $BC = 24.00$ in. Find the radius of the circle.

15. In circle O in Fig. 16-75, central angle $\angle AOB = 96°$ and chord $AB = 18.00$ in. Find the length of the minor arc AB.

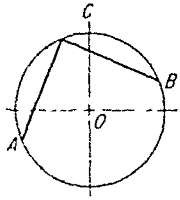


Figure 16-74

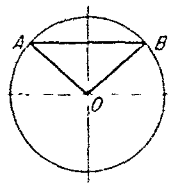


Figure 16-75

16. In circle O in Fig. 16-76, chord $AB = 8$, $DC = 2$, and $OC \perp AB$. Find the radius, OC.

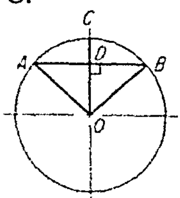


Figure 16-76

17. In Fig. 16-76, $CD = 4$ and $OC = r = 12$. Find AB.

18. Find the length of the common chord AB of the intersecting circles (Fig. 16-77).

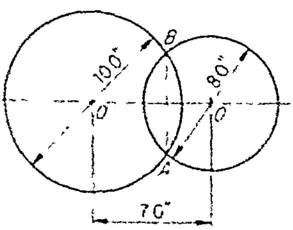


Figure 16-77

19. In triangle ABC (Fig 16 78), $\angle B = 130°$, $\angle C = 20°$, and $AB = 15\ 00$ in Find the altitude, h

20 In Fig 16 79, circle O has a radius, r, of 7 5 in and $\angle AOB = 120°$ Find the area of the indicated segment

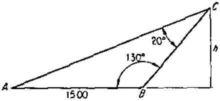


Figure 16-78

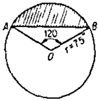


Figure 16-79

Solve the given equations for the interval $0 \leq \theta \leq 2\pi$

21 $\sec^2 \theta - 3 \sec \theta + 2 = 0$

22 $6 \sin \theta - 2 = 3 \sin \theta + 1$

23 $5 \cos \theta - \sqrt{3} = 2 \cos \theta + \dfrac{\sqrt{3}}{2}$

24 $\cos^2 \theta = \sin 2\theta - \sin^2 \theta$

25 $\sec^2 \theta = 2 \tan \theta$

CHAPTER 17

Complex Numbers

An expression of the form $a + bi$ or $a + bj$ is called a *complex number*, where a and b are real numbers and $i = j = \sqrt{-1}$ is an imaginary unit. If $a = 0$, the complex number reduces to a *pure imaginary number*, $0 + 3i$. If $b = 0$, the complex number represents a *real number*, $5 + 0i = 5$.

Two complex numbers are equal only if their **real terms** are **equal** and their **imaginary terms** are **equal**. If $a + bi = c + di$, then it must follow that $a = c$ and $b = d$.

If $a + bi = 0$, it follows that $a = 0$ and $b = 0$.

Complex numbers that are equal except for the signs of the imaginary parts are called *conjugate complex numbers*, $6 + 7i$ and $6 - 7i$.

In carrying out the fundamental algebraic operations involving complex numbers i (or j), assume the same conditions as any real factor.

17-1 ADDITION AND SUBTRACTION

Complex numbers can be added or subtracted by combining, algebraically, the real parts and the imaginary parts, respectively. The sum or difference should appear in the standard form of a complex number, $a + bi$.

EXAMPLE 17-A:

Add the following complex numbers: $3 + 7i$, $-2 - 3i$, $4 - 8i$.

Solution:

Place complex numbers in a column such that the real numbers are in one column and the imaginary numbers in a second column and combine the respective numbers algebraically.

$$\begin{array}{r} 3 + 7i \\ -2 - 3i \\ 4 - 8i \\ \hline 5 - 4i \end{array}$$

which is the sum of the given complex numbers.

EXAMPLE 17-B

Simplify the expression to the form $a + bi$

$$\sqrt{48} + \sqrt{-48} + \sqrt{12} - \sqrt{-12}$$

Solution

Simplify radicals first

$\sqrt{48} = \sqrt{16 \cdot 3} = 4\sqrt{3}$ and $12 = \sqrt{4 \cdot 3} = 2\sqrt{3}$, which are real numbers

$\sqrt{-48} = \sqrt{(-1)(48)} = i\sqrt{48} = 4i\sqrt{3}$ and $\sqrt{-12} = i\sqrt{12} = 2i\sqrt{3}$, which are imaginary numbers

Thus,

$$\sqrt{48} + \sqrt{-48} + \sqrt{12} - \sqrt{-12}$$
$$= (4\sqrt{3} + 2\sqrt{3}) + (4i\sqrt{3} - 2i\sqrt{3}) = 6\sqrt{3} + 2i\sqrt{3}$$

EXAMPLE 17 C

Solve the equation $x + 3yi + 2x + 7yi = 2(3 - 10i)$ for real values of x and y

Solution

Combine like terms

$$x + 3yi + 2x + 7yi = 2(3 - 10i)$$
$$(x + 2x) + (3yi + 7yi) = 6 - 20i \text{ or } 3x + 10yi = 6 - 20i$$

If two complex numbers are equal, their real parts are equal and their imaginary parts are equal. Thus,

$$3x = 6 \text{ and } x = 2,\ 10y = -20 \text{ and } y = -2$$

Hence, the solution of the given equation is $x = 2$ and $y = -2$ The solution can be checked in the usual manner

$$x + 3yi + 2x + 7yi = 2 + 3(-2)i + 2(2) + 7(-2)i$$
$$= 2 - 6i + 4 - 14i = 6 - 20i = 2(3 - 10i)$$

EXERCISES 17-1

Arrange expressions 1-6 in the standard form of the complex number $a + bi$, $(a + bj)$

1. $2i - 15 + 17 + i$

2. $-\sqrt{9} + \sqrt{-9}$

3. $\sqrt{-72} - \sqrt{-72} - \sqrt{-72} + 2\sqrt{32}$

4. $\sqrt{75} - \sqrt{27} - 2\sqrt{3} + 4i + \sqrt{-16}$

5. $j\sqrt{-16} + \sqrt{-1}$

6. $(2 - i)^2$

Simplify and reduce to standard form.

7. $(3 + 2i) + (5 - 3i)$ 8. $(4i - 3) - (3 - 4i)$

9. $(2 - 7i) + (2 + 7i)$ 10. $(7i - 2) + (7i + 2)$

11. $(\sqrt{16} - \sqrt{2}\,i) - (\sqrt{25} + \sqrt{2}\,i)$

12. $(a + bj) - 2(a + bj) + 3(\sqrt{a^2} + j\sqrt{b^2})$

13. $(2 + \sqrt{-16}) + (a - i\sqrt{-16})$

14. $3(x + 2yi) - 2(3x + yi)$

15. $(4 - 5j) - 2(3j) - 3(\sqrt{12})$

16. $(4 - \sqrt{-81}) - (2 + \sqrt{-121}) + (3 - \sqrt{-169})$

17. $(-\sqrt{169} + \sqrt{-169}) - (\sqrt{225} - \sqrt{225})$

18. $(\sqrt{-16} + \sqrt{-49} + \sqrt{-100}) + (-\sqrt{-64} + \sqrt{-9})$

Solve for real values of x and y.

19. $3x + 5yi = 3 + 5i$

20. $-2(4x - yi) = -2(-\sqrt{25} - i\sqrt{25})$

21. $6x - 7yj = 8 - \sqrt{64}$

22. $i^2(3x - 2yi) = (3x - 2yi) + 4(\sqrt{9} - i\sqrt{4i^2})$

23. $(2xj - 3y) = (15 - \sqrt{-12})$

24. $\sqrt{7} - \sqrt{-7} = y\sqrt{-28} - x\sqrt{28}$

25. $4x^2 - 5y^2j = (\sqrt{81} - \sqrt{-25})$

The joint impedance of a series circuit can be determined by the formula $Z_t = R + j(X_L - X_C)$. Find Z_t if:

26. $R = 12$ ohms, $X_L = 6$ ohms, $X_C = 3$ ohms

27. $R = 15$ ohms, $X_L = 8$ ohms, $X_C = 8$ ohms

28. $R = 10$ ohms, $X_L = 0$, $X_C = 10$ ohms

Find R if:

29. $Z_t = 20 + 5j$, $X_L = 12$, $X_C = 7$

30. $Z_t = 9 - 3j$, $X_L = 12$, $X_C = 15$

17-2 MULTIPLICATION AND DIVISION

Complex numbers can be combined by *multiplication* according to those algebraic procedures already adopted for other factors.

$(a + bi)(c + di) = ac + bci + adi + bdi^2$, but $i^2 = -1$

Thus $(a + bi)(c + di) = ac + (bc + ad)i - bd = (ac - bd) + (bc + ad)i$,

where the expression $(ac - bd)$ represents the real portion of the product, and $(bc + ad)i$ represents the imaginary term

EXAMPLE 17 D

Multiply and reduce to standard form $(3 - 4i)(5 + 2i)$

Solution

Every term of the first expression $(3 - 4i)$ must be multiplied by every term of the second factor $(5 + 2i)$ Thus

$$\begin{aligned}(3 - 4i)(5 + 2i) &= (3)(5) + (-4i)(5) + (3)(2i) + (-4i)(2i) \\ &= 15 - 20i + 6i - 8i^2 = 15 - 14i + 8 \\ &= 23 - 14i\end{aligned}$$

is the product in standard form

EXAMPLE 17 E

Solve for real values of x and y if $(x + yi)^2 = 12 - 5i$

Solution

First, expand the term $(x + yi)^2$

$$(x + yi)^2 = x^2 + 2xyi + y^2i^2 = x^2 - y^2 + 2xyi$$

Thus,

$$(x^2 - y^2) + 2xyi = 12 - 5i$$

If two complex numbers are equal, the real parts are equal and the imaginary portions must also be equal, hence, $x^2 - y^2 = 12$ and $2xy = -5$ Next solve the second equation for y and substitute accordingly in the first equation This leads to

$$y = \frac{-5}{2x} \text{ and } x^2 - \left(\frac{-5}{2x}\right)^2 = 12, \text{ or } x^2 - \frac{25}{4x^2} - 12 = 0$$

Simplify further by multiplying through by $4x^2$

$$4x^2\left(x^2 - \frac{25}{4x^2} - 12\right) = 4x^4 - 48x^2 - 25 = 0$$

Factor and solve for x

$$(2x^2 + 1)(2x^2 - 25) = 0$$

where

$$2x^2 + 1 = 0$$

and

$$2x^2 - 25 = 0$$

Therefore,

$$2x^2 = -1 \text{ or } x^2 = -\frac{1}{2}$$

(which will not produce a real root and is disregarded), and

$$2x^2 = 25, \text{ where } x^2 = \frac{25}{2} \text{ and } x = \pm\frac{5}{\sqrt{2}} = \pm\frac{5\sqrt{2}}{2}$$

The solution is completed by solving for y, where

$$y = \frac{-5}{2x} \text{ and } x = \pm\frac{5\sqrt{2}}{2}$$

For $\quad x = \dfrac{5\sqrt{2}}{2}, y = \dfrac{-5}{2x} = \dfrac{-5}{2\left(\frac{5\sqrt{2}}{2}\right)} = \dfrac{-1}{\sqrt{2}} = \dfrac{-\sqrt{2}}{2}$

For $\quad x = \dfrac{-5\sqrt{2}}{2}, y = \dfrac{-5}{2x} = \dfrac{-5}{2\left(\frac{-5\sqrt{2}}{2}\right)} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

Thus the real values for the equation $(x + yi)^2 = 12 - 5i$ are:

$$x = \frac{5\sqrt{2}}{2}, y = \frac{-\sqrt{2}}{2}, \text{ and } x = \frac{-5\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$$

Checking one set $\left(x = \dfrac{5\sqrt{2}}{2}, y = \dfrac{-\sqrt{2}}{2}\right)$

$$\begin{aligned}
(x + yi)^2 &= 12 - 5i \\
(x + yi)^2 &= \left(\frac{5\sqrt{2}}{2} + \frac{-\sqrt{2}}{2}\,i\right)^2 \\
&= \frac{25\cdot 2}{4} + 2\left(\frac{5\sqrt{2}}{2}\right)\left(\frac{-\sqrt{2}}{2}\right)i + \frac{2i^2}{4} \\
&= \frac{50}{4} - \frac{2}{4} - \frac{20i}{4} = \frac{48}{4} - 5i = 12 - 5i
\end{aligned}$$

and conditions are satisfied.

The *quotient of two complex numbers* can be expressed in standard form by *multiplying* both numerator and denominator of the given quotient by the *conjugate of the denominator*. This is analagous to the procedure used in rationalizing denominators containing radicals (Sec. 9-4).

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac+bci-adi-bdi^2}{c^2-di^2}$$
$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

where $c^2 + d^2$ is a real number other than zero

EXAMPLE 17-F

Simplify the given expression to the form $a + bi$

$$\frac{3-4i}{4-3i}$$

Solution

Multiply both numerator and denominator by $(4 + 3i)$, which is the conjugate of $(4 - 3i)$ Collect like terms and reduce to simplest form

$$\frac{3-4i}{4-3i} = \frac{(3-4i)(4+3i)}{(4-3i)(4+3i)} = \frac{12-16i+9i-12i^2}{16-9i^2}$$
$$= \frac{(12+12)+(-16i+9i)}{16+9} = \frac{24-7i}{25}$$
$$= \frac{24}{25} - \frac{7}{25}i, \text{ which is of the form } a+bi$$

EXERCISES 17-2

Simplify to standard form of a complex number

1. $3(2 - 5j)$

2. $3i(2 - 5i)$

3. $(4 - i)(4 + i)$

4. $(4 - j)^2$

5. $(4 + i)^2$

6. $(3 - \sqrt{-3})(2 - \sqrt{-2})$

7. $(2x - 3yj)(x + 3yj)$

8. $(a + bi)(2a - 3bi)$

9. $(2 - i)(3 + 2i)(2 - 3i)$

10. $(2 + i^2)(2 - i^2)$

11. $\dfrac{2+3i}{2-3i}$

12. $\dfrac{4-7i}{3+5i}$

13. $\dfrac{3-\sqrt{-12}}{4+\sqrt{-12}}$

14. $\dfrac{3+2i}{3i}$

15. $\dfrac{9}{4-5i}$

16. $(16 + \sqrt{-18}) - (4 - \sqrt{-2})$

17. $\dfrac{(3-2i)^2}{3+2i}$

18. $2 - \dfrac{4+3i}{3-2i}$

19. $\dfrac{4-i}{2+5i}$

20 $3 + 4i[(i - 3i)^2 - (1 + 3i)^2]$

Solve for real values of x and y.

21. $(x - 3) + (2x - y + 1)i = 6 - 5i$

22. $(x^2 + y^2) + 2yj = 25 - 8j$

23. $(x - iy)^2 = 24 - 10i$

24. $(x - 3yi)^2 = 8 + 6i$

Find the square roots of:

25. $5 + 12i$

26. $24 - 32i$

27. $32i$

28. $5 - i^2$

29. $i^2 - 2\sqrt{-10} - 8$

30. $i^3 - 3i^2 + 5i$

The total or joint impedance of a certain parallel circuit can be expressed by the formula $Z_t = Z_1Z_2/(Z_1 + Z_2)$

31. Find Z_t if $Z_1 = 5 + 12j$ ohms and $Z_2 = 5 - 12j$ ohms.

32. Find Z_t if $Z_1 = 8 + 7j$ and $Z_2 = 12 + 9j$

33. Find Z_t if $Z_1 = -3 - 5j$ and $Z_2 = -5 - 4j$

34. If $Z_t = -7 - j$, and $Z_2 = 4 - 3j$, find Z_1

The formula for the total impedance of a particular series-parallel circuit is $Z_t = Z_s + Z_1Z_2/(Z_1 + Z_2)$

35. Find Z_t if joint series impedance is $Z_s = 6 - 2j$ ohms, $Z_1 = 4 + 3j$ ohms, and $Z_2 = 1 - 2j$ ohms.

36. Find Z_s if $Z_t = 12 - 8j$, $Z_1 = 2 - j$, and $Z_2 = 2 + 3j$

17-3 GRAPHICAL REPRESENTATION

Complex numbers can be represented graphically on a *complex plane*, which is very similar to the rectangular coordinate plane. The *complex plane* is formed by the intersection, at right angles, of a vertical line and a horizontal line. These lines represent the axes of the complex plane and their point of intersection defines the origin, O. The *horizontal axis*, designated as $X'X$, represents the units of *real numbers*, whereas the *vertical axis*, $Y'Y$, represents the *pure imaginary numbers*. Thus, for the complex number $a + bi$, a is plotted in terms of corresponding units along $X'X$ and b is located with reference to units along $Y'Y$. The units of measure along the axes are equal and the principle for representing positive and negative quantities is the same as that established for the rectangular system.

EXAMPLE 17-G:

Represent, graphically, the following complex numbers (Fig. 17-1):

$P_1(3 + 5i)$, $P_2(3 - 5i)$, $P_3(-2 + 4i)$, $P_4(-2 - 4i)$, $P_5(8)$, $P_6(-8i)$, and $P_7(-3 - 5i)$.

Complex numbers can be *added and subtracted graphically* by extending the concept of the *Parallelogram Law* (Sec. 16-6 b and Ex. 16-R).

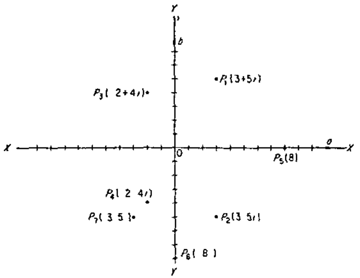


Figure 17-1

EXAMPLE 17-H

Add $5 + 4i$ and $2 - 6i$ graphically (Fig 17-2)

Solution

Plot the given points on a complex plane $P_1(5 + 4i)$ and $P_2(2 - 6i)$

From the origin O, draw the radius vectors OP_1 and OP_2 Construct a line through P_1 parallel to OP_2, in the direction of OP_2 From P_2, construct a line parallel to OP_1 intersecting the first line at P The figure OP_1PP_2 is a parallelogram and the coordinates of the vertex P represents the sum of $5 + 4i$ and $2 - 6i$ Graphically, this appears to be $7 - 2i$

The vectors OP_1, OP, and OP_2 are also used to define complex numbers and other engineering concepts associated with this type of notation

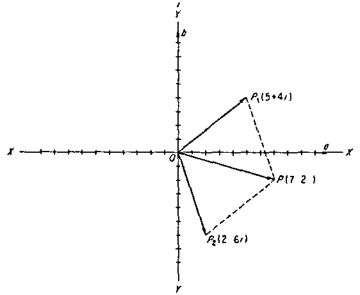


Figure 17-2

EXAMPLE 17-1·

Subtract, graphically, $2 - 6i$ from $5 + 4i$

Solution:

Plot the minuend, $5 + 4i$. Next, change the signs of the complex number defining the subtrahend $(2 - 6i)$, which under these conditions becomes $-2 + 6i$. Plot this point $(-2 + 6i)$ and construct a parallelogram, as in Ex. 17-H. The vertex P completes the parallelogram OP_2PP_1 and represents the difference of the given complex numbers (Fig. 17-3).

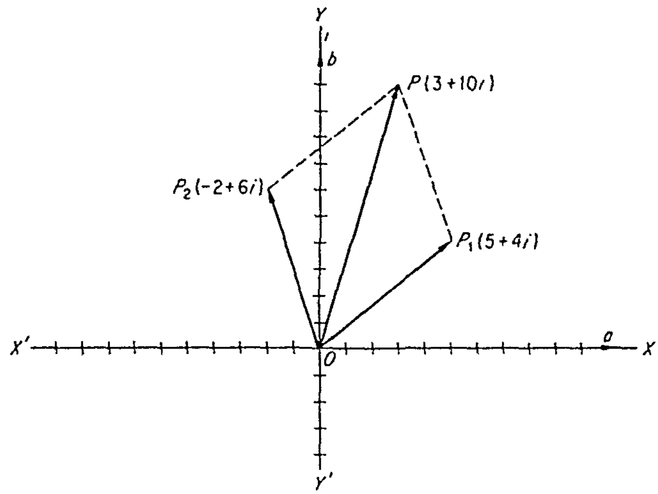


Figure 17-3

From the graph, it appears that P defines the complex number $3 + 10i$.

Another method of obtaining the same results is to plot the points as they appear in the original statement and draw the required vectors accordingly. Before completing the parallelogram, however, construct another vector, equal and opposite to the vector representing the subtrahend. Using this new vector and the vertex representing the minuend, complete the parallelogram (Fig. 17-4).

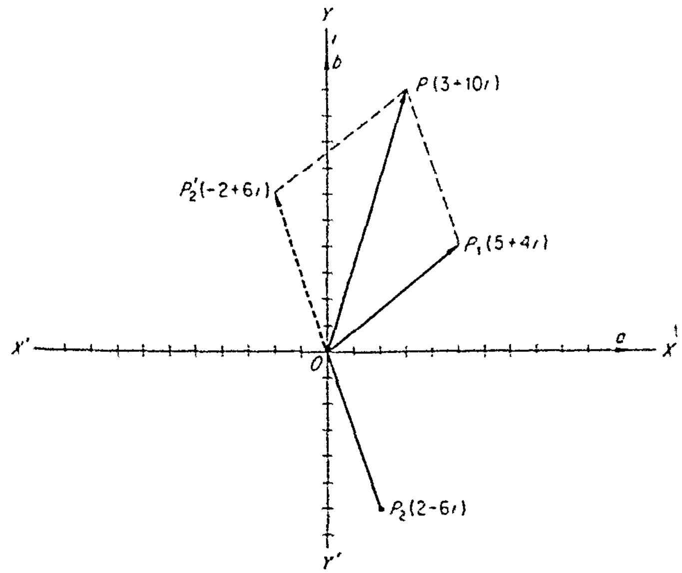


Figure 17-4

The vertex P represents the complex number $3 + 10i$, which is the difference of the two given numbers

This method of representing complex numbers can be used in conjunction with electrical circuits involving inductance, capacitance, resistance, and the resulting impedance Actually, the vector OP_1 (Fig 17-2) could very well represent the impedance, Z, of a circuit with resistance $R = 5$ ohms and an inductive reactance $X_L = j4$ ohms Furthermore, OP_2 could define Z with capacitive reactance $X_C = j6$ ohms and resistance $R = 2$ ohms (reproduced in Fig 17-5)

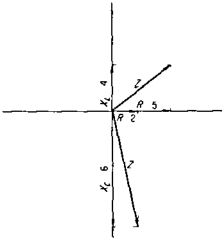


Figure 17-5

Three or more complex numbers can be added or subtracted by first combining any two numbers and then adding to this sum a third number This process is repeated until all numbers are accomodated

EXERCISES 17 3

Represent the complex numbers and their conjugates, graphically

1. $5 - 2i$

2. $-3 + 7i$

3. $-4 - 8i$

4 $\sqrt{8} - \sqrt{-8}$

5. $-2i^2$

6. $\sqrt{-16} - 4i$

7. $-\sqrt{25}$

8. $0 + 0i$

9. $\frac{7}{2} - \frac{9}{2}i$

10. $\frac{2 + i}{2 - i}$

Add or subtract, graphically Check results algebraically

11. $(4 + 3i) + (4 + 5i)$

12. $(5 - 2i) + (6 - 3i)$

13. $(-5 - 5i) + (5 - 5i)$

14. $(3 - 2i) - (2 - i)$

15. $(-7 + 3i) - (-5 + 4i)$

16. $6 + (6 + 5i)$

17. $6 - (6 + 5i)$

18. $(8 - 9i) + (8 + 9i)$

19. $(7 - 2i) + (7 - 2i)$

20 $(10 - 5i) - (10 + 5i)$

21. $(-6 + 8i) - (-6 + 8i)$

22. $(6) - (6i)$

23. $(2 + 3i) + (3 - 4i) + (-5 + 6i)$

24. $(3i - 2) + (-5i + 7) - (3 - 2i)$

17-4 POLAR, OR TRIGONOMETRIC, FORM

The complex number $a + bi$, along with the expression $Z = 3 - 4j$ ohms, has been defined algebraically and represented graphically. Neither concept, initially, seems to provide a satisfactory association with quantity or magnitude as interpreted by such familiar statements as 8.0 gal, 35 amps, 2.54 cm., and the like. To add this dimension requires introduction of another form of a complex number called the **polar**, or **trigonometric, form.**

The complex number $a + bi$, is represented graphically by the point P in Fig. 17-6.

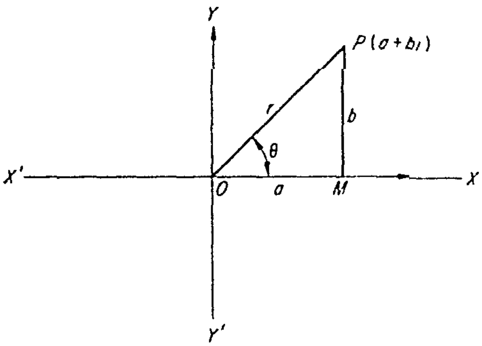


Figure 17-6

A line from the origin O to the point P is equal to the radius vector r. This line forms an angle θ with the $X'X$-axis. Furthermore, the distance from P to the $X'X$-axis is $PM = b$, and the distance from P to the $Y'Y$-axis is $OM = a$. The length r is called the *absolute value*, or the *modulus*, of the complex number $a + bi$, and the *angle* θ is referred to as the *amplitude*, or *argument*, of $a + bi$. Thus,

$$r = |a + bi| = \sqrt{a^2 + b^2}$$

which indicates that the **absolute value**, or **modulus**, of a **complex number** is a **positive real number**. From Fig. 17-6 it can readily be established that:

$r = \sqrt{a^2 + b^2}$; $\tan \theta = b/a$; $\cos \theta = a/r$, where $a = r \cos \theta$ and $\sin \theta = b/r$, where $b = r \sin \theta$, from which:
$a + bi = r \cos \theta + i\, r \sin \theta$, or $a + bi = r(\cos \theta + i \sin \theta)$, where $a + bi$, is called the *algebraic*, or *rectangular, form* of a complex number and $r(\cos \theta + i \sin \theta)$ is called the *trigonometric, or polar, form* of $a + bi$.

EXAMPLE 17-J:

This concept can be further demonstrated in terms of an impedance circuit for which $Z = R - jX_C$. The symbol j is referred to as the *j-operator*

and indicates that the vector associated with this prefix forms an angle of 90° with another vector or some axis of reference Here, X_C forms an angle of 90° with R (Fig 17-7) ($-j$ means that the angle is in the fourth quadrant, whereas $+j$ places the angle in the first quadrant)

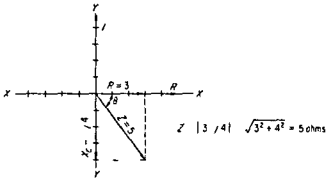


Figure 17-7

To complete the illustration, let $R = 3$ ohms and $X_C = 4$ ohms

The resistance R is plotted along the $X'X$-axis and X_C is plotted along the $Y'Y$-axis A radius vector Z is drawn to the point P, which represents $3 - j4$ The value of Z is identified graphically and from this, its absolute value can be determined Using the relationship

$$r = |a + bi| = \sqrt{a^2 + b^2}$$

it follows that

$$Z = |3 - j4| = \sqrt{(3)^2 + (4)^2} = \sqrt{25}, \text{ or } Z = 5 \text{ ohms}$$

(Note that the $X'X$-axis can be designated as the R-axis and the $Y'Y$-axis as the j-axis)

The process just illustrated is also referred to as *vector addition* These quantities "operate" on vectors, hence, they are *j-operators*

The expressions $3 - 4i$, $3 - 4j$, or $3 - j4$ all identify the same mathematical concept This concept is sometimes associated with a scientific principle, which means that its appearance may be altered to fit the given conditions Its basic definition, however, remains the same

EXAMPLE 17 K

Represent the complex number $12\sqrt{3} + 12i$ in trigonometric form

Solution:

Plot the point and sketch the corresponding triangle OMP (Fig 17-8)

With the use of Fig 17-8 and other information, r and θ can be determined

$$r = |a + bi| = \sqrt{a^2 + b^2}$$

where $a = 12\sqrt{3}$ and $b = 12$

Thus,

$$r = \sqrt{(12\sqrt{3})^2 + (12)^2} = \sqrt{432 + 144} = \sqrt{576} = 24$$

Furthermore,

$$\tan\theta = \frac{b}{a} = \frac{12}{12\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.57735,$$

where $\theta = 30°$. Therefore, the trigonometric, or polar, form of $12\sqrt{3} + 12i$ is $24(\cos 30° + i \sin 30°)$, which is sometimes written as $24\,\underline{/30°}$.

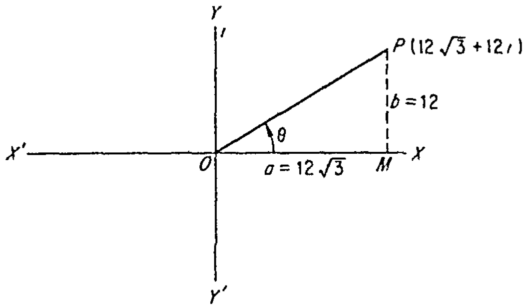


Figure 17-8

EXAMPLE 17-L:

Write the complex number $10(\cos 135° + i \sin 135°)$ in rectangular form.

Solution:

With the $X'X$-axis as the initial side and the origin O as the vertex, construct an angle of 135° with the terminal side extended. From O, measure off 10 units on the terminal side, locating P. Thus, P represents the given complex number, where $OP = r = 10$ (Fig. 17-9).

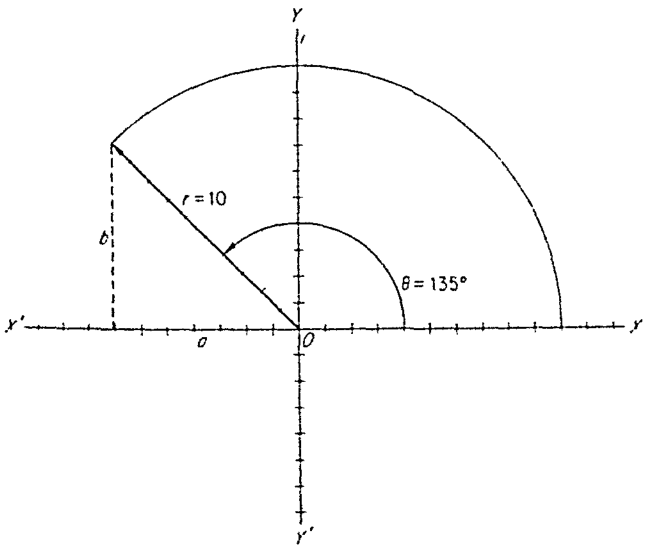


Figure 17-9

Recall that $a = r\cos\theta$ and $b = r\sin\theta$
Furthermore,

$$\cos 135^\circ = -\cos(180^\circ - 135^\circ) = -\cos 45^\circ = -0.70711$$
$$\sin 135^\circ = \sin(180^\circ - 135^\circ) = \sin 45^\circ = 0.70711$$

Thus,

$$a = 10(-0.70711) = -7.0711$$
$$b = 10(0.70711) = 7.7011$$

Therefore, the rectangular form of $10(\cos 135^\circ + i\sin 135^\circ)$ is

$$-7.0711 + 7.0711i$$

EXERCISES 17-4

Represent the following complex numbers in polar, or trigonometric, form. Use the smallest value of the amplitude that will meet the given conditions. Take advantage of a sketch.

1. $4\sqrt{3} + 4i$ *2.* $7 + 7i$

3. $-3 + 3\sqrt{3}i$ *4.* $-2\sqrt{2} - 2i\sqrt{2}$

5. $\dfrac{\sqrt{3}}{3} - i$ *6.* 15

7. $15i$ *8.* $-6i$

Determine the absolute value of Z.

9. $Z = 6 + 8i$ *10.* $-5 + 12i = Z$

11. $Z = 24 - 7i$ *12.* $1 - i = Z$

Represent each of the complex numbers in rectangular form.

13. $6(\cos 45^\circ + i\sin 45^\circ)$ *14.* $5(\cos 90^\circ + i\sin 90^\circ)$

15. $4(\cos 150^\circ + i\sin 150^\circ)$ *16.* $10(\cos 240^\circ + i\sin 240^\circ)$

17. $\frac{3}{8}(\cos 315^\circ + i\sin 315^\circ)$ *18.* $100(\cos 70^\circ + i\sin 70^\circ)$

19. $20(\cos 160^\circ + i\sin 160^\circ)$ *20.* $5(\cos 180^\circ + i\sin 180^\circ)$

REVIEW EXERCISES 17-5

Find the real values of a and b that satisfy the given equations (Ex. 1-4).

1. $a + bi = i\sqrt{36} - \sqrt{36} + \sqrt{-36}$

2. $(a + bi)^2 = 9 - 12i$

3. $(a + b)^2 + 2bi = 36 - 10i$

4. $(2a^2 - b^2) + (2a + b)i = 34 + 6i$

Find the square roots of each complex number (Ex. 5-8).

5. $9 + 40i$

6. $-27 - 36i$

7. $2i$

8. $-1 + 2i\sqrt{6}$

Simplify to standard form, $a + bi$.

9. $(i^3 - 3i^2 + 3i - 1)$

10. $(2 - i)(4 - 4i + i^2)$

11. $\dfrac{1}{3 - i} - \dfrac{1}{3 + i}$

12. $\sqrt{-2}(\sqrt{2} - i\sqrt{2} + \sqrt{-2})$

13. $\dfrac{3i - 2}{2 - 3i}$

14. $2 - \dfrac{3i - 2}{2 - 3i}$

15. $(\sqrt{3} + \sqrt{-3})(\sqrt{3} - \sqrt{-3})$

16. $(2 - i)^3$

Add or subtract, graphically.

17. $(4i + 4) + (-4i + 4)$

18. $(6 + 12i) - (12 - 6i)$

19. $10(\cos 45° + i \sin 45°) + 12(\cos 135° + i \sin 135°)$

20. $8(\cos 270° + i \sin 270°) + 20(\cos 60° + i \sin 60°)$

Given: $Z_t = R_t \pm jX = Z(\cos\theta + j\sin\theta)$. Find Z_t for the various conditions.

21. $Z_t = 50(\cos 50° + j \sin 50°)$

22. $Z_t = 60(\cos 300° + j \sin 300°)$

23. $Z = 100(\cos 90° + j \sin 90°)$

24. $Z_t = 21 + 28j$

25. $Z_t = 15 - 36j$

CHAPTER 18

Analytic Geometry

The application of algebra to the various concepts of geometry is called *analytic geometry*. Nearly all geometric forms or figures utilized in technology can be represented by an algebraic expression. Invariably, if a condition can be illustrated geometrically (graphically), that same relationship can be expressed analytically (algebraically). This section will touch on some of the fundamental principles of analytic geometry as they may involve the work of the technician. The technician may be required to find the center of an arc of a cam, design parabolic reflectors for use in pick-up or transmission of various electronic signals, study the effect of loading on long-span structural arches, and so on.

18-1 EQUATIONS OF STRAIGHT LINES

18-1a SLOPE. A straight line can be determined by two points or by the slope and a point. *Slope of a line is defined as the tangent of the angle of inclination, where the angle is measured with respect to the horizontal, or x-axis.* This angle varies between 0° and 180°. In mathematics, the slope of a line is designated by the letter m (Fig. 18-1).

In civil technology, especially in laying out highways, slope is referred to as *grade*. Grade is expressed in terms of percentages and is defined as the ratio of *rise* to *run* (Fig. 18-2).

By definition, the *slope* of a line can be expressed as the ratio of the difference of ordinates to the difference of abscissas (Fig. 18-3).

Hence, the slope of a straight line can be determined by the equation

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

where the values of x and y must be taken in the same order.

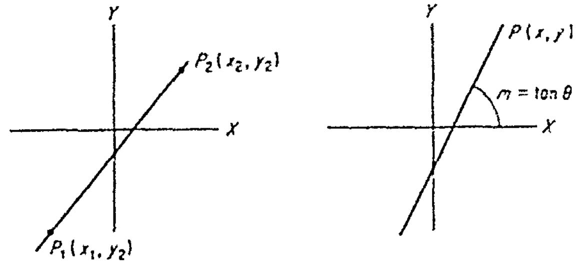
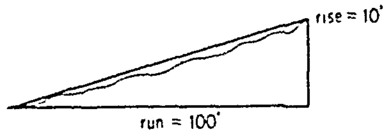
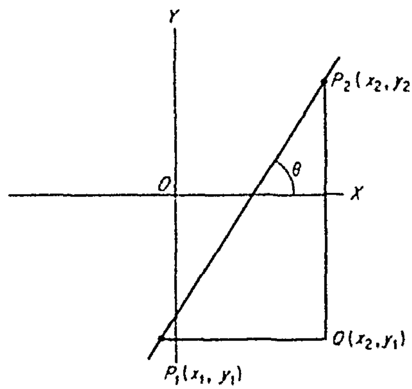


Figure 18-1



$$\text{Grade} = \frac{\text{rise (feet)}}{\text{run (feet)}} \times 100 = \frac{10\ \text{ft}}{100\ \text{ft}} \times 100 = 10\%$$

Figure 18-2

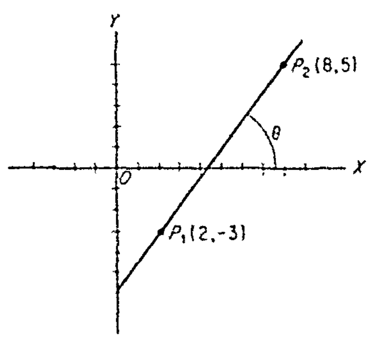


$$m = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

Figure 18-3

EXAMPLE 18-A:

Find the slope of the straight line passing through the points (8, 5) and (2, −3) (Fig. 18-4).



$$m = \tan\theta$$

Figure 18-4

Solution:

Let the point (8, 5) be designated as P_2 with coordinates $x_2 = 8$ and $y_2 = 5$, and let point (2, −3) be designated as P_1 with coordinates $x_1 = 2$ and $y_1 = -3$.

Substituting accordingly,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-(-3)}{8-2} = \frac{5+3}{6} = \frac{8}{6} = \frac{4}{3}$$

Thus, the slope of the straight line passing through (8, 5) and (2, −3) is equal to $\frac{4}{3}$

The angle of inclination of a line whose slope is given can be determined by referring to the table of natural functions

Hence, $\tan\theta = m$, and $\theta = \arctan m$ In this example,

$$\tan\theta = \tfrac{4}{3} = 1\ 3333, \text{ and } \theta = \arctan 1\ 3333 = 53°8$$

EXAMPLE 18 B

Find the slope of the straight line in Fig 18 5 that contains the points (−4, 7) and (5, −2)

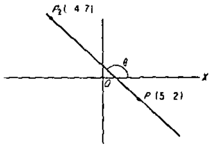


Figure 18 5

Solution

The selection of coordinates to represent P_1 and P_2 is arbitrary as long as the order is maintained throughout the given discussion Here, P_2 (−4, 7) and P_1 (5, −2)

Thus,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-(-2)}{-4-5} = \frac{7+2}{-9} = \frac{9}{-9} = -1$$

A negative slope indicates that $90° < \theta \leq 180°$, whereas a positive slope indicates that $0° \leq \theta < 90°$ Furthermore, $m = 0$ represents a horizontal line (Fig 18 6)

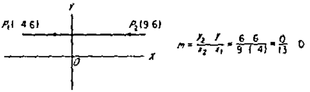
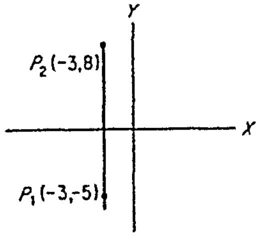


Figure 18-6

On the other hand, the slope of a vertical line is said to be indeterminate (Fig 18-7)



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-(-5)}{-3-(-3)} = \frac{13}{0}$$

(division by zero leads to an indeterminate quotient)

Figure 18-7

EXERCISES 18-1

Find the slope of the straight line passing through the indicated points (Ex. 1-10).

1. (5, 5) and (1, 1)

2. (−4, 12) and (1, 3)

3. (−11, 9) and (−2, 1)

4. (−7, −1) and (3, −6)

5. (3, 6) and (−7, −12)

6. (7, 8) and (0, 0)

7. $\left(-5, \frac{1}{2}\right)$ and $\left(14, \frac{1}{2}\right)$

8. (0, 0) and (13, 0)

9. (2, 11) and (2, 0)

10. (10, 10) and (−10, −10)

Find the slope of a line (Ex. 11-14):

11. forming an angle of 45° with the x-axis.

12. parallel to the line passing through (4, 6) and (0, 0).

13. forming an angle of 102° with the horizontal.

14. whose angle of inclination is 0°.

15. Find the grade of a highway that has a rise of 135 ft in a run of 1,000 ft.

16. The grade of a road is 35%. Find the rise per 100-ft run.

17. Given a line forming an angle of 60° with the x-axis passing through (6, 6). Find another point on the line.

18. Construct a line passing through (0, 5) and parallel to a line with $m = \frac{3}{4}$ containing the point (−4, −7).

19. Construct the straight line with:
(a) $m = \frac{5}{2}$ passing through the origin.
(b) $m = -\frac{2}{5}$ passing through the origin.
From a sketch representing the two lines of 19a and 19b, estimate the angle formed by the intersection of the lines.

20. What is the slope and angle of inclination of a line with a grade of 30%?

18-1b Point-Slope Form: A straight line can also be defined if the slope of the line is known and a point on the line is given. A slight modification of the slope formula leads to an equation of a straight line known as the

point-slope form This formula can be used to find an equation of a straight line when both the slope and a point are known

$$m = \frac{y - y_1}{x - x_1} \text{ or } y - y_1 = m(x - x_1)$$

where x_1 and y_1 are the coordinates of the given point and m is the slope

EXAMPLE 18-C

Determine the equation of a straight line passing through the point (7, 3) with a slope of $\frac{2}{3}$

Solution

$$m = \frac{2}{3},\ x_1 = 7, \text{ and } y_1 = 3$$

Substituting, respectively, in the point-slope form leads to the equation of the given line It should be pointed out that there is only one straight line that meets these conditions

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{2}{3}(x - 7)$$

or

$$3y - 9 = 2x - 14$$

which leads to the equation of the line $2x - 3y = 5$

Checking: If $2x - 3y = 5$ contains the point (7, 3) then these coordinates must satisfy the given equation

$$2(7) - 3(3) = 5$$

$$14 - 9 = 5,$$

and the equation is balanced

The point-slope form can be compared to the equation for thermal expansion of a solid

$$l - l_1 = k(t - t_1)$$

where l represents the final length of the material, l_1 the original length, t the final temperature, t_1 the original temperature, and k a constant that is equal to the product of the coefficient of linear expansion, α, and the original length, l_1 $\alpha = 6.5 \times 10^{-6}$ for steel (change in length per degree change °F)

EXAMPLE 18-D.

A steel tie rod measures 250 in when the temperature is 50°F Find the length of the rod when the temperature reaches 100°F

Solution:

$l_1 = 250$ in., $t = 100°F$, $t_1 = 50°F$, $\alpha = 6.5 \times 10^{-6}$, and $k = \alpha \times l_1 = (6.5 \times 10^{-6})(250) = 16.25 \times 10^{-4}$
Substituting accordingly,

$$l - l_1 = k(t - t_1)$$
$$l - 250 = (6.5 \times 10^{-6} \times 250)(100 - 50)$$
$$l - 250 = (16.25 \times 10^{-4})(50) = 8.1 \times 10^{-2}$$

Thus,

$$l = 250 + 0.081 = 250.081 \text{ in.}$$

This relationship can be plotted where k represents the slope and (t_1, l_1) the given point (Fig. 18-8).

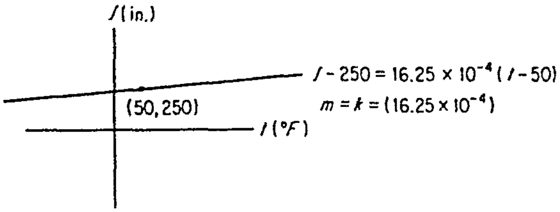


Figure 18-8

18-1c Slope-Intercept Form: It is often convenient to define a straight line in terms of its intercepts. The *intercepts* of a line are the points at which the line crosses the respective axes. The *x-intercept* (point where the line crosses the x-axis) is designated by the letter a and is designated as $(a, 0)$, whereas the *y-intercept* is designated as $(0, b)$ (Fig. 18-9).

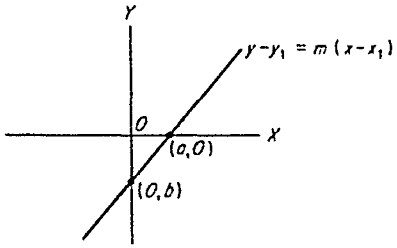


Figure 18-9

If the y-intercept $(0, b)$ is substituted into the point-slope form, the following equation of a straight line emerges: $y - y_1 = m(x - x_1)$, where $x_1 = 0$ and $y_1 = b$. Thus, $y - b = m(x - 0)$ and $y - b = mx$, or $y = mx + b$, which is known as the slope-intercept form of a straight line. This equation is useful when the slope and y-intercept are known.

A very familiar example of the slope-intercept form is the formula for converting Centigrade temperature readings to Fahrenheit readings ($°F = \frac{9}{5}°C + 32°$) (Fig. 18-10).

The slope-intercept form proves expedient when need arises for a quick sketch of a straight line. This is accomplished by converting the given

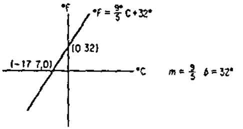


Figure 18-10

equation into the slope intercept form, whereby the slope and y-intercept (point) are recognized

EXAMPLE 18 E

Sketch the line whose equation is $5x - 3y = 18$

Solution

Convert the equation of the given line into the slope-intercept form by rearranging terms and solving for y in terms of x

$$5x - 3y = 18, \text{ or } 3y = 5x - 18$$

Furthermore,

$$y = \frac{5}{3}x - \frac{18}{3} = \frac{5}{3}x - 6$$

Thus the equation $5x - 3y = 18$ now appears in slope-intercept form, $y = \frac{5}{3}x + (-6)$, where the slope, m is $\frac{5}{3}$ and the y-intercept, b, is -6 Since a point and the slope are known, the straight line is defined and can be sketched (Fig 18-11)

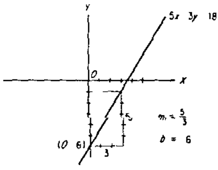


Figure 18-11

18-1d TWO-POINT FORM If the definition of slope, $m = y_2 - y_1/x_2 - x_1$, is substituted into the point-slope form, another equation for a straight line is developed This equation is referred to as the two-point form and is used when two points (of a straight line) are given

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y - y_1 = m(x - x_1)$$

Substituting for m,

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

which leads to:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \text{ the two-point form}$$

EXAMPLE 18-F:

Find the equation of a straight line determined by the points (6, 9) and (−2, 3) (Fig. 18-12).

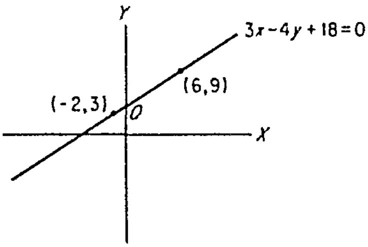


Figure 18-12

Solution:

The selection of points to be designated as P_1 (x_1, y_1) and P_2 (x_2, y_2) is merely one of preference. Once the decision is made, however, the order must remain consistent. In this example, both options will be demonstrated.

In the first illustration, let (6, 9) be designated as P_2 (x_2, y_2) and (−2, 3) be designated as P_1 (x_1, y_1). Substituting into the two-point form:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

where $x_1 = -2$, $x_2 = 6$, $y_1 = 3$, and $y_2 = 9$

$$\frac{y - 3}{x - (-2)} = \frac{9 - 3}{6 - (-2)} = \frac{6}{8} = \frac{3}{4}$$

Furthermore,

$$4(y - 3) = 3(x + 2)$$

$$4y - 12 = 3x + 6$$

and

$$3x - 4y + 18 = 0$$

which is the equation of the line containing the points (6, 9) and (−2, 3).

Alternate Solution:

Let (6, 9) be designated as P_1 and (−2, 3) as P_2. Then, $x_1 = 6$, $x_2 = -2$, $y_1 = 9$, and $y_2 = 3$.

Substituting,

$$\frac{y-9}{x-6}=\frac{3-9}{-2-6}=\frac{-6}{-8}=\frac{3}{4}$$

Simplifying further,

$$4(y-9)=3(x-6)$$

$$4y-36=3x-18$$

and again,

$$3x-4y+18=0$$

This should serve as a check and suggest that *two points define one and only one straight line*

18-1e INTERCEPT-FORM A most convenient way of plotting a straight line is to find the x intercept and the y-intercept and draw a line through these two unique points The x-intercept is designated by the point $(a, 0)$ and the y-intercept by the point $(0, b)$ (Fig 18-13)

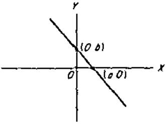


Figure 18-13

By substituting the points $(a, 0)$ and $(0, b)$ into the two point form, another equation representing a straight line is developed, called the intercept form

Let $x_2 = 0$, $x_1 = a$, $y_2 = b$, and $y_1 = 0$

It then follows that $y - 0/x - a = b - 0/0 - a$, or $y/x - a = b/-a$, from which $x/a + y/b = 1$

This form is useful when the equation of a line is required, given the intercepts It is not, however, the most convenient form to obtain Usually several steps are required to convert the given equation into the intercept form A more direct way of finding intercept is to set $x = 0$ and solve the given equation for y, and then set $y = 0$ and solve for the x-intercept

EXAMPLE 18 G

If the x-intercept is equal to 6 and the y-intercept is equal to -4, find the equation of the straight line defined by these conditions (Fig 18-14)

Solution:

Substituting $a = 6$ and $b = -4$ into the intercept form leads to the equation

$$\frac{x}{6}+\frac{y}{-4}=1$$

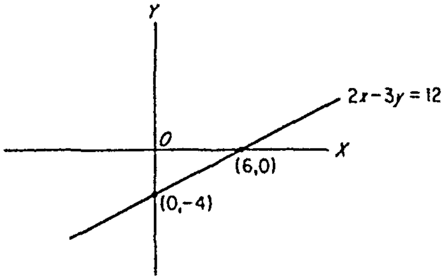


Figure 18-14

Simplifying,

$$4x - 6y = 24, \text{ or } 2x - 3y = 12$$

which is the equation of a straight line with intercepts $a = 6$ and $b = -4$.

The various forms representing equations of straight lines, as determined in this section, are summarized below (Fig. 18-15).

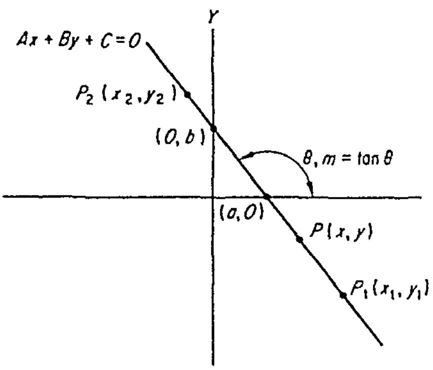


Figure 18-15

General Form:	$Ax + By + C = 0$
Point-Slope Form:	$y - y_1 = m(x - x_1)$
Slope-Intercept Form:	$y = mx + b$
Two-Point Form:	$\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$
Intercept Form:	$\dfrac{x}{a} + \dfrac{y}{b} = 1$

EXERCISES 18-2

Express the following equations in slope-intercept form. Indicate the value of the slope and y-intercept (Ex. 1-6).

1. $x + 6y = 24$
2. $3x - 5y = 15$
3. $7x + 14y + 29 = 0$
4. $3x - 5y = 0$
5. $3x = 20$
6. $6y - 11 = 0$

Find the intercepts of the following straight lines, if they exist (Ex. 7-18).

7. $2x - 3y = 12$
8. $2x - 3y - 12 = 0$

9. $3y + 2x = 12$

10. $2x = 3y$

11. $2x = 13$

12. $3y = 13$

13. $4x + 5y - 20 = 0$

14. $5x - 4y - 20 = 0$

15. $4x = 0$

16. $5y = 0$

17. $3x - 14y = 0$

18. $14y + 3x = 24$

Determine the equations of the straight lines defined by the following conditions

19. passing through (0, 0) and (5, 6)
20. passing through (−3, 4) and (4, −3)
21. passing through (12, 0) and (0, −6)
22. passing through (5, 7) and (−12, 7)
23. passing through (0, 0) and (8, 0)
24. passing through (0, 0) and (0, −9)
25. x-intercept = 3 and y-intercept = 3
26. x-intercept = 0 and y-intercept = −5
27. x-intercept = −11 and y-intercept = −11
28. $a = 4$ and $b = 0$
29. y-intercept = 4 and slope = $\frac{7}{2}$
30. $b = -4$ and $m = \frac{2}{7}$
31. $b = 5$, $m = -4$
32. passing through (2, 5), $m = \frac{9}{-10}$
33. passing through (7, −3), $m = -\frac{3}{5}$
34. passing through (0, 0), $m = 1$
35. passing through (3, 6), $m = 0$
36. passing through the origin with $m = 0$
37. passing through (1, 2) and parallel to the line $4x - 3y = 36$
38. parallel to a line containing points (4, 2) and (−3, −5)
39. passing through (−12, 0) and parallel to the y-axis
40. Determine if all of the given points lie on a straight line (12, 4), (5, 2) and (−2, 0) Justify your conclusion
41. Determine if the following points lie on a straight line (−7, 5), (0, 3), and (7, 0)
42. Determine if the point (−4, −1) lies on the line $3x - 5y + 7 = 0$
43. Find a point that lies on the line $2x + 7y = 42$

Exercises 44-46 involve construction.

44. Construct a line passing through $(5, 0)$ and parallel to $x - 5y = 6$.

45. Construct a line bisecting the line segment with end points $(6, 4)$ and $(-1, 1)$.

46. Construct a line with $m = -\frac{7}{5}$ and $b = -\frac{7}{5}$.

47. A brass rod measures 70 in. when the temperature is 30°F. Find the length of the rod when the temperature reaches 120°F. The coefficient of linear expansion for brass is $\alpha = 9.2 \times 10^{-6}$.

48. Find the length of the rod in exercise 47 when the temperature drops from 30°F to −30°F.

18-2 DISTANCE AND MID-POINT FORMULAS

The distance between two given points, whose coordinates are known, can be determined by applying the Pythagorean Theorem (Fig. 18-16).

$$d^2 = (P_1P_2)^2 = (P_1Q)^2 + (QP_2)^2$$

but,

$$QP_2 = y_2 - y_1, \text{ and } P_1Q = x_2 - x_1$$

Thus,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

which is called the *distance formula*. Again the order of coordinates and the respective signs are consistent.

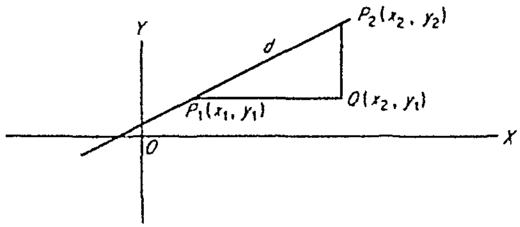


Figure 18-16

EXAMPLE 18-H:

Find the distance between the two points in Fig. 18-17 whose coordinates are (10, 9) and (−2, 4).

Solution:

Let $x_2 = 10$, $y_2 = 9$, $x_1 = -2$, and $y_1 = 4$. Substituting in the distance formula, accordingly:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{[10 - (-2)]^2 + [9 - 4]^2} = \sqrt{(12)^2 + (5)^2}$$
$$= \sqrt{144 + 25} = \sqrt{169} = 13$$

Notice that if the selection of P_1 and P_2 had been reversed, the results would not be affected

Let $x_2 = -2$, $y_2 = 4$, $x_1 = 10$, and $y_1 = 9$
$$d = \sqrt{(-2 - 10)^2 + (4 - 9)^2} = \sqrt{(-12)^2 + (-5)^2}$$
$$= \sqrt{144 + 25} = \sqrt{169} = 13$$

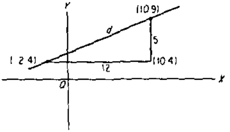


Figure 18-17

Mid-Point Formula.

As an exercise in analysis, the technician is asked to derive the *Mid-Point Formula*. This formula is used to obtain the coordinates of a point that bisects a line segment whose end points are known (Fig 18-18). Hint: parallel lines cut off equal segments

$$x = \frac{1}{2}(x_1 + x_2), \text{ and } y = \frac{1}{2}(y_1 + y_2)$$

These two equations are known as the mid-point formulas, in which (x, y) are the coordinates of a point midway between P_1P_2

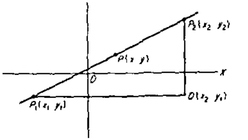


Figure 18-18

EXERCISES 18-3

Find the distance between the following sets of points.

1. (9, 12) and (0, 0)

2. (−1, 3) and (4, 15)

3. (21, −1) and (−3, 6)

4. (−3, −5) and (3, 3)

5. (0, 12) and (3, 12)

6. (0, 0) and (−12, 0)

7. (0, −5) and (−5, 7)

8. (6, 6) and (−6, −6)

9. (−5, 5) and (5, −5)

10. (0, 0) and (0, −5)

Let the coordinates of Ex. 1-10 represent end-points of line segments and determine the respective mid-points (Ex. 11-20).

18-3 ANGLE BETWEEN TWO LINES

18-3a PARALLEL LINES: If *two lines are parallel, their slopes are equal.* Thus, if m_1 is the slope of line l_1, and m_2, the slope of line l_2, $m_1 = m_2$. The most convenient way of determining if two lines are parallel is to convert the respective equations into the slope-intercept form.

EXAMPLE 18-I:

Determine if the two lines $3x + 5y = 12$ and $6x + 10y = 7$ are parallel.

Solution:

Convert each equation into the slope-intercept form and compare the slopes. If the slopes are equal, the lines are parallel.

$$3x + 5y = 12, \text{ or } 5y = -3x + 12$$

from which,

$$y = -\frac{3}{5}x + \frac{12}{5}, \text{ and } m_1 = -\frac{3}{5}$$

$$6x + 10y = 7, \text{ or } 10y = -6x + 7$$

from which,

$$y = -\frac{6}{10}x + \frac{7}{10} = -\frac{3}{5}x + \frac{7}{10} \text{ and } m_2 = -\frac{3}{5}$$

Since $m_1 = m_2 = -\frac{3}{5}$, the lines are parallel.

18-3b PERPENDICULAR LINES: *If two lines are perpendicular, the product of their slopes is equal to* -1. Thus, $m_1 m_2 = -1$, or $m_1 = -1/m_2$.

EXAMPLE 18-J:

Determine if the lines $3x + 4y = 24$ and $4x - 3y = 12$ are perpendicular to each other (Fig. 18-19).

Solution:

Find the slope of each line and compare accordingly.

$$3x + 4y = 24, \text{ or } 4y = -3x + 24$$

and

$$y = -\frac{3}{4}x + 6, \text{ where } m_1 = -\frac{3}{4}$$

Furthermore,

$$4x - 3y = 12, \text{ or } 3y = 4x - 12$$

and

$$y = \frac{4}{3}x - 4, \text{ where } m_2 = \frac{4}{3}$$

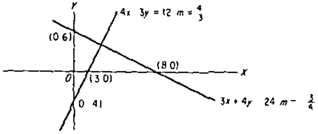


Figure 18-19

If the two lines are perpendicular, the product of the slopes must be equal to -1, or $m_1 m_2 = -1$
Hence,

$$m_1 = -\frac{3}{4} \text{ and } m_2 = \frac{4}{3}, \text{ and } m_1 m_2 = \left(-\frac{3}{4}\right)\left(\frac{4}{3}\right) = -1$$

Thus, the lines are perpendicular Notice also that $-\frac{3}{4}$ is the negative reciprocal of $\frac{4}{3}$, or $-\frac{3}{4} = -1/\frac{4}{3}$

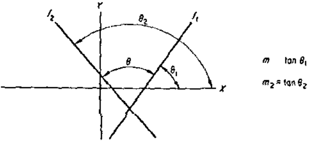


Figure 18-20

18-3c ANGLE BETWEEN TWO STRAIGHT LINES The angle between two straight lines can be determined by substituting the respective slopes into the trigonometric identity. tangent of the difference of two angles (Fig 18-20)

$$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$$

By definition,

$$m_2 = \tan\theta_2 \text{ and } \tan\theta_1 = m_1$$

Hence,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}$$

which is known as the formula for the angle between two straight lines.

In this formula, m_2 should represent the slope of the line that forms the larger angle of inclination. A sketch of the lines will be extremely useful and in most cases will suggest the selection of m_1 and m_2. Recall that the tangent function is negative for angles that lie between 90° and 180°; furthermore, $\tan\theta = -\tan(180° - \theta)$.

EXAMPLE 18-K:

Find the angle formed by the lines whose equations are $3x - 5y = 15$ and $4x + y + 6 = 0$.

Solution:

The first step suggests converting the equations into the slope-intercept form followed by a sketch (Fig. 18-21).

$$3x - 5y = 15 \text{ leads to } y = \frac{3}{5}x - 3$$

$$4x + y = 6 \text{ leads to } y = -4x + 6$$

Let $m_2 = -4$ and $m_1 = \dfrac{3}{5}$

Thus,

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{-4 - \frac{3}{5}}{1 + (-4)\left(\frac{3}{5}\right)} = \frac{\frac{-20-3}{5}}{\frac{5-12}{5}} = \frac{23}{7}$$

Therefore,

$$\theta = \text{arc tan}\,\frac{23}{7} = \text{arc tan } 3.2857 = 73°4'$$

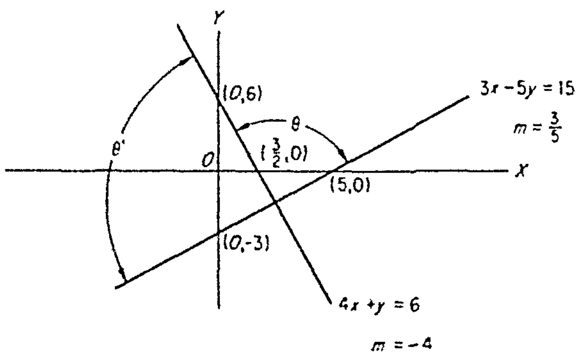


Figure 18-21

Alternate Solution

For exploratory purposes, the values of m_2 and m_1 will be interchanged, hence, $m_2 = \frac{3}{5}$ and $m_1 = -4$
Making the appropriate substitutions leads to

$$\tan \theta' = \frac{\frac{3}{5} - (-4)}{1 + \left(\frac{3}{5}\right)(-4)} = \frac{\frac{3}{5} + 4}{1 - \frac{12}{5}} = \frac{\frac{23}{5}}{-\frac{7}{5}} = -3\ 2857$$

and $\theta = 106°56'$ Notice that the angles are supplementary

$$\theta + \theta = 73°4' + 106°56 = 180°$$

EXERCISES 18-4

From the following straight lines, select the equations that represent (a) parallel lines and (b) lines that are perpendicular

1. $2x - 7y = 21$	*2.* $7y - 26x = 71$
3. $x - 5y = 12$	*4* $5y + 4x = 20$
5. $4x - 14y = 17$	*6.* $6x + 20y = 79$
7. $5x + y = 30$	*8* $\frac{3}{2}x + \frac{2}{5}y = 11$
9. $5x + 13y = 19$	*10* $12x + 15y - 13 = 0$
11. $5y = 13x$	*12.* $13x - \frac{7}{2}y = 42$
13. $2x = 7y$	*14.* $3x - 10y = 5$
15. $x + 5y = 12$	*16.* $3x + 10y = 5$

Determine the equations of lines that meet the indicated criteria (Ex 17-25)

17. parallel to the line $3x - 2y = 12$ and passing through the point $(0, 0)$

18. containing the point $(5, 4)$ and perpendicular to the line $3x - 2y = 12$

19. containing the point $(-3, -7)$ and parallel to a line passing through $(0, 0)$ with slope $\frac{2}{3}$

20. perpendicular to the line segment with end points $(6, 8)$ and $(-2,2)$ passing through the mid-point of the segment

21. parallel to the line $3x - 7y$ with y-intercept equal to -4

22. perpendicular to the line $3x - 7y$ with y-intercept equal to -4

23. perpendicular to the line $3x - 4y - 0$ and 5 units away from the origin (2 solutions)

24. parallel to the line $3x = 4y$ and 5 units away from the origin (2 solutions)

25. Find the angle between two lines whose slopes are equal, respectively, to:

(a) $\frac{3}{4}$ and 4. (b) $-\frac{3}{4}$ and 4.

Given the following straight lines, find the angle between them.

26. $6x - 3y = 10$, $2x - y = 12$

27. $4x + y = 0$, $3x - 2y = 9$

28. $2y - 3x = 6$, $3x + 2y = 6$

29. $8x + 4y = 15$, $9x - 3y = 16$

30. $8x + 4y = 15$, $9x + 3y = 16$

31. Find the angle between two surveyor's lines with grades of 15% and 75%, respectively.

18-4 DISTANCE-POINT TO LINE

The distance from a point to a line is measured along the perpendicular from the point to the line. This distance can be determined by applying the following equation:

$$d = \frac{A_{x_1} + B_{y_1} + C}{\sqrt{A^2 + B^2}}$$

The equation of the line is $Ax + By + C = 0$ and the coordinates of the point are $P_{(x_1, y_1)}$.

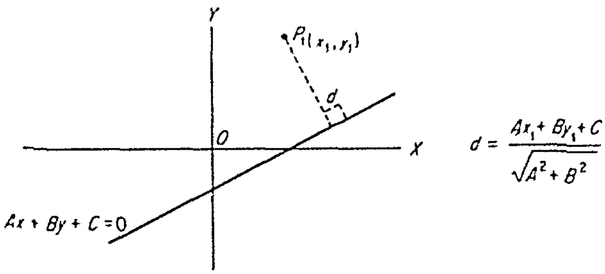


Figure 18-22

EXAMPLE 18-L:

Find the distance from the point $P_{(2,3)}$ to the line $3x + 4y + 12 = 0$.

Solution:

A sketch indicates the following conditions (Fig. 18-23).

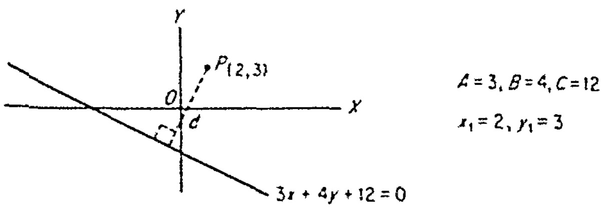


Figure 18-23

Substituting accordingly,

$$d = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} = \frac{3(2) + 4(3) + 12}{\sqrt{(3)^2 + (4)^2}} = \frac{6 + 12 + 12}{\sqrt{25}} = \frac{30}{5} = 6$$

EXAMPLE 18 M

Find the distance from the point $P_{(10\ \ 3)}$ to the line $3x + 4y + 12 = 0$

Solution

First make a sketch followed by the appropriate substitutions

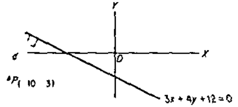


Figure 18 24

$$d = \frac{Ax_1 + By \ + C}{\sqrt{A^2 + B^2}} = \frac{3(-10) + 4(-3) + 12}{\sqrt{(3)^2 + (4)^2}}$$

$$= \frac{-30 - 12 + 12}{\sqrt{25}} = \frac{-30}{5} - -6$$

Note In both examples the distance from the point to the line measured is 6 units however the signs are opposite Algebraically this indicates that the respective points are located on opposite sides of the line Perhaps it might be more appropriate if the formula were to be re written to indicate absolute value, since the number associated with distance is acceptably positive

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

A sketch will usually give a clear indication of the particular conditions

The distance between two parallel lines is the measure of their common perpendicular This distance can be determined by selecting a point on one of the lines and applying the formula

$$d = \left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$$

EXAMPLE 18 N

Find the distance between the two parallel lines $4x - 3y = 24$ and $4x - 3y + 12 = 0$ (Fig 18 25)

Solution

The first step involves selecting a point on one of the lines This is

accomplished by assigning an arbitrary value to either x or y and solving for the remaining variable. A most convenient number is zero. In this illustration, a point on line $4x - 3y = 24$ will be used.

Setting $x = 0$ and solving for y leads to the coordinates of the given point: $4(0) - 3y = 24$, from which $-3y = 24$ and $y = -8$.

The problem now resolves to one of finding the distance from $P_{(0,-8)}$ to the line $4x - 3y - 12 = 0$.

Substituting respectively,

$$d = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}\right| = \left|\frac{4(0) + (-3)(-8) - 12}{\sqrt{(4)^2 + (-3)^2}}\right| = \left|\frac{24 - 12}{\sqrt{25}}\right| = \frac{12}{5}$$

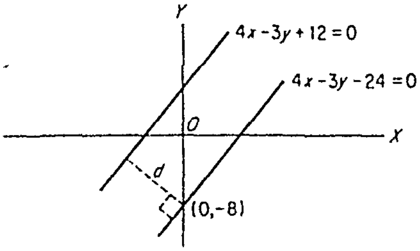


Figure 18-25

EXERCISES 18-5

Find the distance from the given point to the given line (Ex. 1-10).

1. $P_{(0,0)}$ $5x - 12y = 26$

2. $P_{(8,-1)}$ $5x - 12y = 26$

3. $P_{(2,4)}$ $4x + 3y + 20 = 0$

4. $P_{(-2,-4)}$ $4x + 3y + 20 = 0$

5. $P_{(-3,4)}$ $7x - 24y = 25$

6. $P_{(0,0)}$ $2x - 5y = 0$

7. $P_{(3,3)}$ $5x + y = 0$

8. $P_{(-5,-6)}$ $2x - 4y - 5 = 0$

9. $P_{(-7,-8)}$ $x + y + 8 = 0$

10. $P_{(2,-3)}$ $3x - 2y - 12 = 0$

Find the distance between the following pairs of parallel lines (Ex. 11-15).

11. $3x - 4y + 12 = 0$
$3x - 4y - 12 = 0$

12. $12x + 5y = 60$
$12x + 5y = 0$

13. $7x - 24y + 20 = 0$
$7x - 24y - 12 = 0$

14. $2x - 3y = 24$
$6x - 9y = 72$

15. $5x + 7y = 35$
$15x + 21y = 35$

16. There are two lines whose equations are $4x + 3y = 0$ and $3x - 4y = 24$, respectively.

(a) Point (0, 0) lies on the line defined by the equation $4x + 3y = 0$. Find the distance from $P_{(0,0)}$ to the second line.

(b) Find the distance between $P_{(0,0)}$ and the point of intersection of the two lines.

17. Find the equation of a line parallel to $12x - 5y = 60$ and 5 units above it

18. Determine which of the two lines $3x + 4y - 25 = 0$ or $4x - 3y - 25 = 0$ passes closest to the origin $(0, 0)$

19. Find the equation of a line with slope $\frac{4}{3}$ that is 5 units away from the origin $(0, 0)$ (2 solutions)

20. Find the equation of a line $\sqrt{10}$ units from the origin $(0, 0)$ with slope -1 (2 solutions)

18-5 CONICS

A family of curves associated with a plane and a cone, called the *conic sections*, appear repeatedly in the field of technology These curves consist of the *circle*, *parabola* *ellipse*, and *hyperbola* (Fig 18-26) Special cases of the conics include a point and two intersecting straight lines

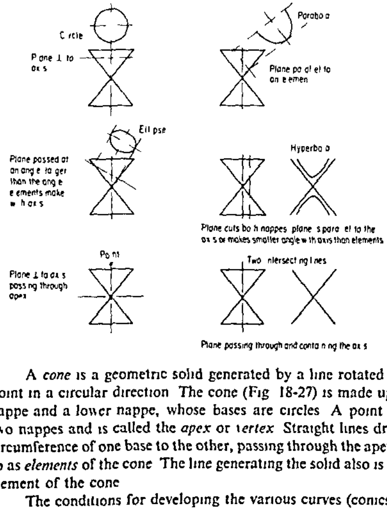


Figure 18-26

A *cone* is a geometric solid generated by a line rotated about a fixed point in a circular direction The cone (Fig 18-27) is made up of an upper nappe and a lower nappe, whose bases are circles A point separates the two nappes and is called the *apex* or *vertex* Straight lines drawn from the circumference of one base to the other, passing through the apex, are referred to as *elements* of the cone The line generating the solid also is considered an element of the cone

The conditions for developing the various curves (conics) are demonstrated in Fig 18-26

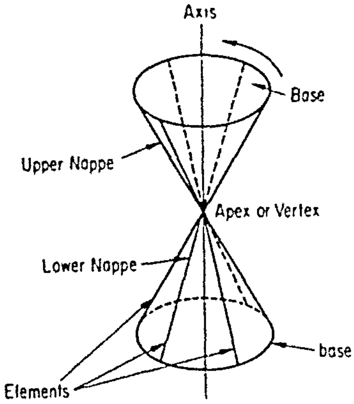


Figure 18-27

The treatment of the conics will involve the circle and parabola only. These curves have far greater immediate application than the hyperbola and ellipse.

18-5a CIRCLE: *A circle is defined as the locus (path) of all points equidistant from a fixed point.* The distance is called the *radius*, and the fixed point is called the *center* of the circle (Fig. 18-28).

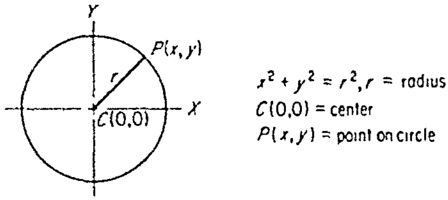


Figure 18-28

The equation of a circle can be developed by applying the distance formula to the definition of a circle.

$$d^2 = (x - x_1)^2 + (y - y_1)^2 \text{ (distance formula)}$$

Let $x_1 = 0$, $y_1 = 0$, and $d = r$ (radius). Thus, $r^2 = (x - 0)^2 + (y - 0)^2 = x^2 + y^2$, or, $x^2 + y^2 = r^2$, is the equation of a circle with center at the origin and radius equal to r.

The equation of a circle whose center is at some point, (h, k), other than $(0, 0)$ can also be derived by applying the definition of a circle and the distance formula (Fig. 18-29).

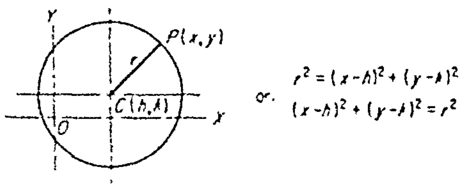


Figure 18-29

EXAMPLE 18-O:

Find the equation of a circle with its center at (7, —4) and its radius equal to 12 (Fig. 18-30).

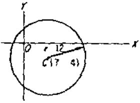


Figure 18-30

Solution

The equation of the circle defined by the given conditions can be determined by substituting accordingly into the standard form

$$(x - h)^2 + (y - k)^2 = r^2$$

where,

$$h = 7,\ k = -4, \text{ and } r = 12$$

$$(x - 7)^2 + [y - (-4)]^2 = (12)^2$$

or

$$(x - 7)^2 + (y + 4)^2 = 144$$

which is the equation of a circle with its center at (7, −4) and a radius of 12

If the expression $(x - 7)^2 + (y + 4)^2 = 144$ (standard form) is expanded it will lead to the equation of the given circle in general form

$$x^2 + y^2 + Dx + Ey + F = 0$$

$$(x - 7)^2 + (y + 4)^2 = 144$$

$$x^2 - 14x + 49 + y^2 + 8y + 16 = 144$$

or

$$x^2 + y^2 - 14x + 8y - 79 = 0$$

where

$$D = -14,\ E = 8, \text{ and } F = -79$$

EXAMPLE 18 P

Given the equation of a circle, $x^2 + y^2 - 6x + 10y - 30 = 0$ find the radius and center

Solution

This process involves completing the square in x and the square in y of the given equation

Thus

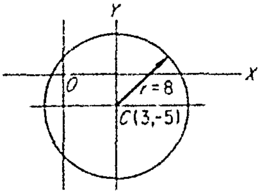
$$x^2 - 6x + \left(\frac{6}{2}\right)^2 + y^2 + 10y + \left(\frac{10}{2}\right)^2 = 30 + \left(\frac{6}{2}\right)^2 + \left(\frac{10}{2}\right)^2$$

and

$$x^2 - 6x + 9 + y^2 + 10y + 25 = 30 + 9 + 25 = 64$$

Factoring yields $(x - 3)^2 + (y + 5)^2 = 64$

Hence, the equation $x^2 - 6x + y^2 + 10y - 30 = 0$ defines a circle with its center at $(3, -5)$ and with a radius of $r = \sqrt{64} = 8$.



$x^2 - 6x + y^2 + 10y - 30 = 0$, and $(x-3)^2 + (y+5)^2 = 64$

are equivalent equations (general form-standard form)

Figure 18-31

A significant item to bear in mind is that the coefficients of x^2 and y^2 be equal to unity (1) before attempting to define the circle. For example, the equation $9x^2 + 9y^2 = 25$ is an equation of a circle with its center at the origin; however, the radius is not equal to $\sqrt{25} = 5$. To find the radius, it becomes necessary to divide through by the coefficients of x^2 and y^2.

$$9x^2 + 9y^2 = 25, \quad \frac{9x^2}{9} + \frac{9y^2}{9} = \frac{25}{9}$$

Furthermore,

$$x^2 + y^2 = \frac{25}{9}, \text{ where } r = \sqrt{\frac{25}{9}} = \frac{5}{3}$$

EXERCISES 18-6

Determine the center and radius of the circles defined by the following equations (Ex. 1-15).

1. $x^2 + y^2 = 1$
2. $x^2 + y^2 = 196$
3. $x^2 + y^2 - 189 = 0$
4. $3x^2 + 3y^2 = 567$
5. $\sqrt{2}x^2 + \sqrt{2}y^2 = 6$
6. $\frac{x^2}{4} + \frac{y^2}{4} = 15$
7. $(7x)^2 + (7y)^2 = 49$
8. $x^2 + y^2 = \frac{1}{4}$
9. $(x - 3)^2 + (y - 2)^2 = 225$
10. $x^2 + (y + 4)^2 = 625$
11. $(x + 10)^2 + y^2 - 40 = 0$
12. $(2x + 16)^2 + 4(y - 7)^2 = 4$
13. $x^2 - 4x + y^2 - 6y = 83$
14. $x^2 + y^2 + 8x - 14y = 191$
15. $5(x - 7)^2 + 5(y - 7)^2 = 415$

Determine the equations of the circles defined by the respective conditions. Use a sketch (Ex. 16-28).

16. $C_{(0,0)}\ r = 7$
17. $C_{(0,0)}\ r = 9$
18. $C_{(0,0)}\ r = \frac{1}{3}$
19. $C_{(0,0)}\ r = \sqrt{7}$
20. $C_{(0,0)}\ r = \frac{\sqrt{3}}{2}$
21. $C_{(2.10, 2.15)}\ r = 2.05$
22. $C_{(-5.5, 7.1)}\ r = 11.2$
23. $C_{(0.0, -6.0)}\ r = \sqrt{10.0}$

24. $C_{(0,0)}P_{(5,-12)}$ 25. $C_{(-2,5)}P_{(-5,9)}$

26. Passing through the points $(-2, 6)$, $(2, 6)$ and tangent to the line $x - y = 6$

27. Tangent to both axes, center on line $x + y = 0$, and $10\sqrt{2}$ units away from the origin

28. Center on the x-axis and passing through the points $(5, 7)$, $(-2, 0)$

Two points determine a straight line, whereas three points determine one and only one circle. Thus, given three points, the equation of a circle can be determined by substituting the given points into the general form $x^2 + y^2 + Dx + Ey + F = 0$ and solving the system. This is referred to as the analytical solution. The geometric (graphical) solution relies on the principle that perpendicular bisectors of chords pass through the center of the circle.

29. Find the equation of the circle passing through the given points
(a) $(0, 0)$, $(25, 5)$, $(12, -8)$ (b) $(3, 4)$, $(-4, 3)$, $(0, 5)$

30. Determine the equation of the circle containing chords whose end points are $(8, -6)$, $(0, -10)$, and $(-10, 0)$, $(6, 8)$, respectively

31. Same as 30, except the end-points of the chords are $(1, 2)$, $(-6, 9)$, and $(-5, 2)$, $(2, 3)$

32. Find the equation of the circle that circumscribes the isosceles triangle whose base is 10.00 in. and whose vertex angle is 40°. Find also the equation of the inscribed circle. (End points of the base are -5.0 and 5.0)

33. Given a triangle with vertices, respectively, at $(-4, 0)$, $(12, 0)$, and $(0, 4\sqrt{3})$, determine the equation of the circumscribing circle along with the equation of the inscribed circle

34. Determine the equation of the circles with their centers at $(0, 0)$ and $(12, 0)$, respectively, that are tangent at the point $(8, 0)$

18-5b PARABOLA With reference to the conics, the parabola can be considered the most natural curve. If a line, chord, or cable with fixed ends is suspended, left hanging freely of its own weight, the curve formed will be a parabola. The path of a projectile approximates a parabola (Fig. 18-32)

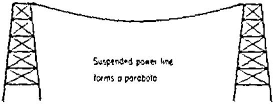


Figure 18-32

In technology, the parabola is useful in the design of various reflectors and electronic antennas along with suspension-type structures.

A parabola is the locus (path) of all points equidistant from a fixed point and a fixed line. The fixed line is called the *directrix*, whereas the fixed point is referred to as the *focus*, *F*. Furthermore, the focus lies on the axis of the

parabola, which is a line perpendicular to the directrix. Along with this, the point where the parabola crosses the axis is called the *vertex*, V. The vertex, V, is mid-way between the directrix, D, and the focus, F. A line perpendicular to the axis passing through the focus will cut off a line segment equal to $2p$. This segment is called the *latus rectum* (Fig. 18-33).

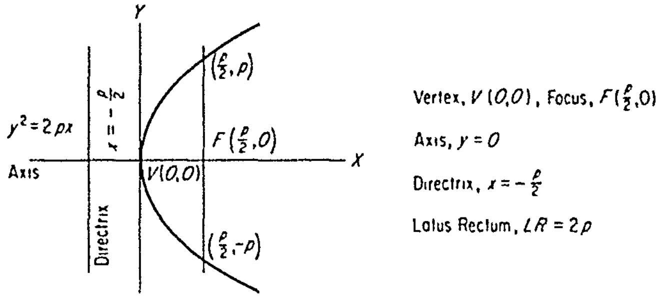


Figure 18-33

The equation $y^2 = 2px$ (standard form) gives the general trend of the curve, whereas the geometric characteristics (focus, vertex, end points of the latus rectum, along with the axis) provide the additional information leading to a quick sketch.

EXAMPLE 18-Q:

The equation of a parabola is $y^2 = 16x$. Determine the coordinates of the focus, vertex, end points of the latus rectum, the equation of the directrix, and sketch the curve.

Solution:

$y^2 = 2px$ is the standard form of a parabola opening to the right with vertex at the origin and axis on the x-axis, or $y = 0$. From $y^2 = 16x$, it follows that:

$$2p = 16, p = 8, \text{ and } \frac{p}{2} = 4$$

Thus, the coordinates of the focus are (4, 0), and the equation of the directrix is $x = -4$.

Furthermore, the length of the latus rectum is equal to $2p$, or $LR = 16$. Hence, the coordinates of the end-points of LR are (4, 8) and (4, -8).

With the several characteristics defined, the curve can be sketched (Fig. 18-34).

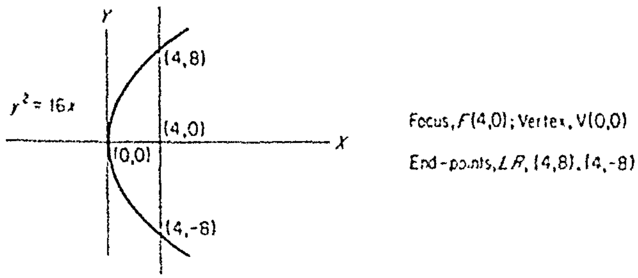


Figure 18-34

The equation $y^2 = 2px$ is referred to as a parabola, opening to the right with vertex at the origin and axis on the x-axis. Three other unique parabolas can be defined that also have vertices at the origin and axis on the coordinate (x, y) axes.

The sign ($\pm$) of the linear term will give the direction the curve will assume. Furthermore, the parabola always opens away from the directrix. The axis is considered the axis of symmetry, and the latus rectum will always be taken as positive. Finally, for the equation to appear in standard form, the coefficient of the second-degree term must be positive (+) and equal to unity (1).

The four cases in which the parabola has its vertex at the origin and axis on the coordinate axes are treated in Fig. 18-35. All equations appear in standard form.

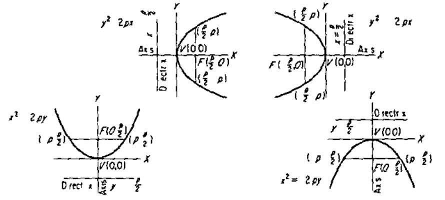


Figure 18-35

EXAMPLE 18.Q

Sketch the parabola $-4x^2 = 27y$. (Locate the vertex, focus, and the end points of the latus rectum.)

Solution

Transpose the given equation to resemble general form.

$$\frac{-4x^2}{-4} = \frac{27y}{-4} \quad \text{or} \quad x^2 = -\frac{27}{4}y$$

This can now be recognized as a parabola (Fig. 18-35) opening downward, with vertex at the origin and axis on the y-axis.

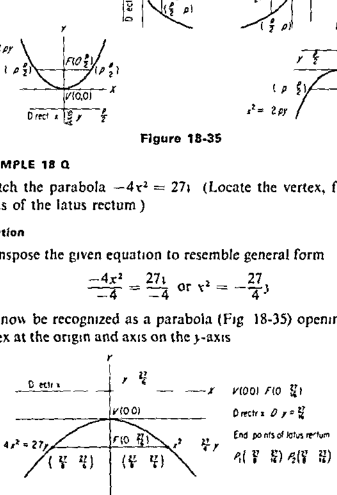


Figure 18-36

Furthermore,

$$2p = \frac{27}{4}, p = \frac{27}{8}, \text{ and } \frac{p}{2} = \frac{27}{16} \quad \left(LR = \frac{27}{4}\right)$$

This seems to provide the basic information needed to sketch the curve (Fig. 18-36).

EXERCISES 18-7

Sketch the various parabolas listed in exercises 1-10. Locate the vertex, focus, and the end points of the latus rectum.

1. $y^2 = 24x$

2. $y^2 = -24x$

3. $x^2 = 24y$

4. $x^2 = -24y$

5. $2x^2 = -19y$

6. $3y^2 = -28x$

7. $-7y^2 = 36x$

8. $-4x^2 = -64y$

9. $x^2 - 4x - 12y - 8 = 0$

10. $10y^2 - 10x = 0$

Determine the equations of the various parabolas defined by the given conditions (Ex. 11-20):

11. vertex (0, 0), focus (3, 0).

12. vertex (0, 0), focus (0, −2).

13. vertex (0, 0), equation of directrix, $y = 6$.

14. focus (0, −4), length of latus rectum = 16, and curve passes through point (0, 0).

15. focus (0, 8), equation of directrix, $y = -8$.

16. directrix, $y = 36$, and curve passes through (0, 0).

17. axis, $y = 0$, directrix contains the point (−6, 6), curve passes through (6, 12).

18. passing through (−8, 2) (8, 2) and directrix, $y = -8$.

19. passing through (−4, 16) (0, 0) with focus (0, −16).

20. passing through (1, 2), (4, 4), and (4, −4).

Find the point(s) of intersection of the following curves (straight lines). Sketch the conditions (Ex. 21-28).

21. $y^2 = 32x, y = 4x$

22. $x^2 + y^2 = 16, 3x - 4y + 12 = 0$

23. $x^2 + y^2 = 9, x^2 = -8y$

24. $x^2 = 16y, x - 4y + 8 = 0$

25. $y^2 = 2x, y^2 = -2x$

26. $x^2 = 24y, y^2 = 24x$

27. $x^2 = -24y, x = -24y$

28. $x^2 + y^2 = 36, x^2 = -36y$

29. Find the equation of a circle passing through the origin with its center at the focus of $y^2 = 48x$

30. Find the equation of a circle, with its center at the origin, passing through the end points of the latus rectum of $x^2 = -32y$

Several scientific and engineering principles can be associated with the properties of a parabola, or perhaps it is better to say that the parabola has certain characteristics that are the basis for various engineering designs and scientific considerations

For a structure with a uniformly distributed load (a bridge closely approximates this condition), cables that take the form of a parabola will support the load most evenly (Fig 18-37)

If a parabola is rotated about its axis (Fig 18-38) a paraboloid is generated

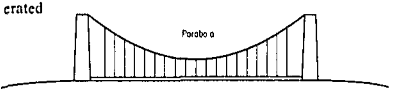


Figure 18-37

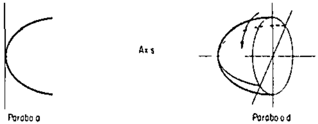


Figure 18-38

If the source of power for radio-sound waves and other impulses is concentrated at the focus, the waves will bounce off of the surface of a paraboloid reflector in (nearly) straight-parallel lines The same is true for receiving various waves The incoming signals will reflect from the surface and be directed to the focus The electro-mechanical receiving equipment will be installed at that point This is the principle on which optical equipment, radar antennas, sound pick-up instruments, and the like, are designed

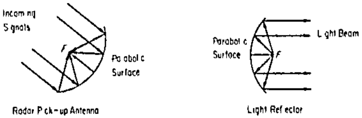


Figure 18-39

Equations describing uniformly accelerated motion and free-falling bodies are parabolic in nature,

$$s = \frac{1}{2}gt^2, \text{ or } t^2 = \frac{2s}{g},$$

where g represents the pull of gravity (32.2 ft/sec²), t is the time of fall (seconds), and s is the distance of fall (feet).

EXERCISES 18-8

1. Given a railroad suspension bridge (Fig. 18-40) with main support cables taking on the form of a parabola defined by the equation $x^2 = 1{,}800y$, find (a) the height of the towers, and (b) the height of the vertical support cables at 50.00-ft intervals.

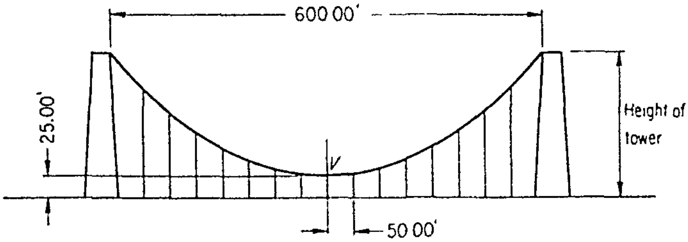


Figure 18-40

2. A line sags of its own weight, as shown in Fig. 18-41. Find the equation of the curve (parabola).

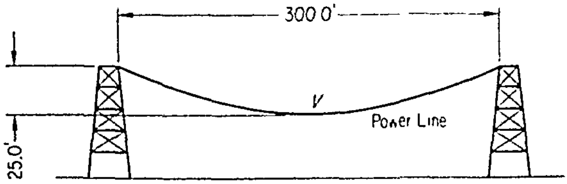


Figure 18-41

The tension developed in the cables of a suspension bridge can be computed by applying the following formulas (Fig. 18-42).

$$T_d = \frac{wl^2}{8d} \qquad T_m = \frac{1}{2}wl\sqrt{1 + \frac{l^2}{16d^2}}$$

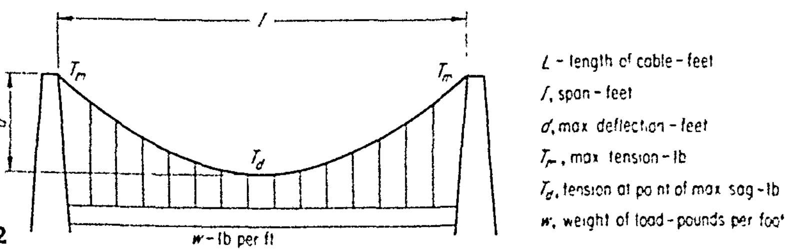


Figure 18-42

3. Find the maximum tension developed in the cable in exercise 1 if it supports a horizontal load of 1,500 lb/ft. Compare this to the tension developed at the vertex of the parabola.

4. If the horizontal load supported by the cable in exercise 1 is 1,200 lb/ft, find the maximum tension developed, along with the tension, at the midpoint of the support.

5. If the cable in exercise 2 weighs 0.25 lb/ft, find the maximum tension developed in the cable.

The length of a cable, L, can be determined by the equation

$$L = l\left[1 + \frac{8}{3}\left(\frac{d}{l}\right)^2\right] \quad \text{(this is a design approximation)}$$

6. Find the length of the cable for each of the exercises above for which this equation is applicable (1, 2,)

7. A power line sags of its own weights, as shown in Fig 18-43 Find the original length

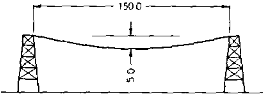


Figure 18-43

8. If an object is dropped freely from a point 3 000 0 ft above the ground, how long will it take to reach the ground? (This will only be an approximation because of resistance caused by atmospheric conditions)

9. A vehicle accelerates uniformly at the rate of 30 0 ft/sec² How long will it take the vehicle to travel 600 0 ft?

10. Find the deflection of a suspended cable whose length is 1,030 ft and the distance between the supports equals 1,000 ft

18-6 OPTIONAL

18-6a ELLIPSE In addition to the special cases of a point and two intersecting straight lines the ellipse and hyperbola also belong to the family of curves called the conics These two curves will not be of immediate concern to the technician because of their limited application A brief study will be made of them, however, to round out the topic

An ellipse is the locus of all points where the sum of the distances from two fixed points is constant The two fixed points are called the *foci* and lie on the major axis The length of the *major axis* is usually designated by the arbitrary constant $2a$, which also happens to represent '*the sum of the distances from two fixed points,*' mentioned previously in the definition Furthermore, the points at which the ellipse crosses the major axis are called the *vertices* The vertices are also the end points of the major axis

Besides the major axis, the ellipse has a minor axis The *minor axis* is perpendicular to the major axis at a point midway between the foci, which is the same as the mid-point of the major axis This point of intersection of the set of axes is called the *center* of the ellipse The length of the *minor axis* is usually designated by another arbitrary constant, $2b$, and the distance between the foci is defined as $2c$ The ellipse is symetrical about both of these axes Finally, lines passing through the foci perpendicular to the major-axis cut off segments (*latus rectum*) equal to $LR = 2b^2/a$

Figure 18-44a is a sketch of an ellipse with the major axis on the x-axis the minor axis on the y-axis, and its center at the origin (0, 0), whereas Fig. 18-44b sketches an ellipse with its major axis on the y axis and its minor axis on the x-axis and its center at the origin. In both sketches, other meaningful properties are also defined.

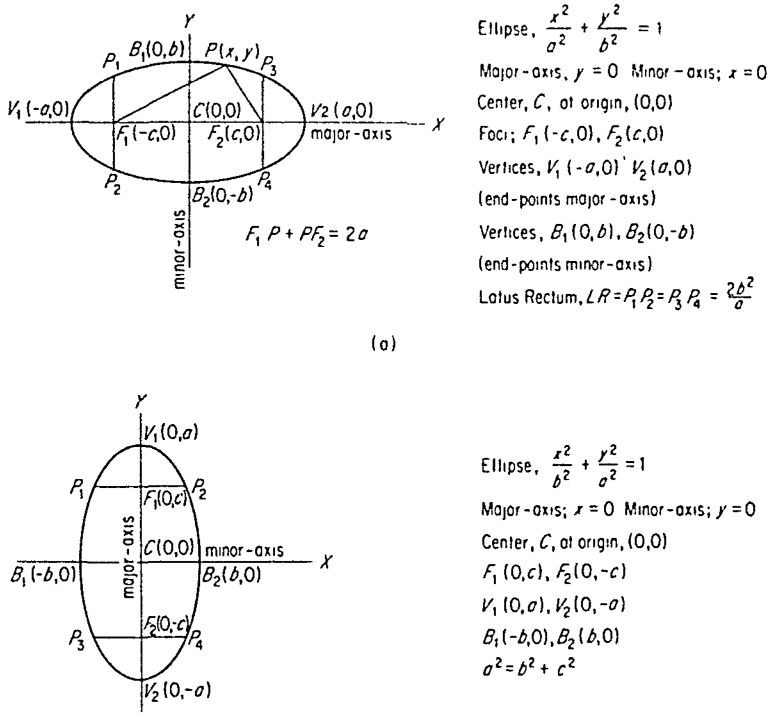


Figure 18-44 (b)

The length of the major axis, $2a$, is always longer than the minor axis, $2b$; consequently, $a^2 > b^2$. These values, a and b, define immediately the position of the major and minor axes.

For example, in the equation $x^2/16 + y^2/9 = 1$, $a^2 = 16$ and $b^2 = 9$; thus, the major axis will be on the x-axis. On the other hand, $5x^2/2 + 5y^2/3 = 1$ represents an ellipse with the major axis on the y-axis, since $\frac{3}{5} > \frac{2}{5}$. Notice that the last equation had to be transposed into standard form, $\frac{x^2}{\frac{2}{5}} + \frac{y^2}{\frac{3}{5}} = 1.$

Notice that the second-degree term with the largest denominator defines the major axis. If the denominator of x^2 is larger than the denominator of y^2, the major axis lies on the x-axis, or on a line parallel to the x-axis. Likewise, if the denominator of the y^2 term is larger than the denominator of the x^2 term, the major axis will be on the y-axis or on a line parallel to the y-axis.

EXAMPLE 18-R:

Sketch the ellipse defined by the equation $\frac{x^2}{169} + \frac{y^2}{25} = 1.$

Solution

Since the equation is already in standard form (Fig 18-44a), the major axis and center can be determined by inspection

Thus, $a^2 = 169$, $b^2 = 25$, where $169 > 25$, which indicates that the major-axis lies on the x-axis

Furthermore,

$$a = \sqrt{169} = \pm 13, \text{ and } 2a = 26$$

$$b = \sqrt{25} = \pm 5, \text{ and } 2b = 10$$

$$c = \sqrt{a^2 - b^2} = \sqrt{169 - 25} = \sqrt{144} = \pm 12, \text{ and } 2c = 24$$

With these values established, the coordinates associated with the various characteristics of the ellipse can now be determined

Thus,

Foci $F_1(-12, 0)$, $F_2(12, 0)$

Vertices $V_1(-13, 0)$, $V_2(13, 0)$

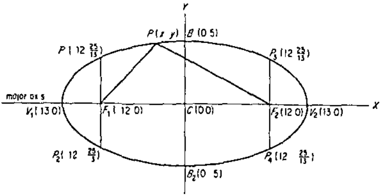
$B_1(0, 5)$, $B_2(0, -5)$

Latus Rectum $LR = \dfrac{2b^2}{a} = \dfrac{2(25)}{13} = \dfrac{50}{13}$

End Points of LR $P_1\left(-12, \dfrac{25}{13}\right)$, $P_2\left(-12, -\dfrac{25}{13}\right)$

$P_3\left(12, \dfrac{25}{13}\right)$, $P_4\left(12, -\dfrac{25}{13}\right)$

Sufficient information is now available to make a reasonable sketch (Fig 18-45)



$\frac{x^2}{169} + \frac{y^2}{25} = 1$ $F_1P + PF_2$ 26

Figure 18 45

The end points of the latus rectum should satisfy the equation, as a check, P_4 $(12, -\frac{25}{13})$ will be used $x = 12$, $y = -\frac{25}{13}$

Substituting in the equation leads to

$$\frac{(12)^2}{169} + \frac{\left(-\frac{25}{13}\right)^2}{25} = 1, \frac{144}{169} + \frac{\frac{625}{169}}{25} = 1$$

Continuing:

$$\frac{144}{169} + \frac{625}{(25)(169)} = \frac{144}{169} + \frac{25}{169} = \frac{169}{169} = 1 \quad \text{(equation is satisfied)}$$

The principle of "whispering-halls" is based on the shell of an ellipsoid along with the foci. If a source of light or sound is projected from a focus, it will be reflected to the other focus. The second focus is the only point that will receive the signal (Fig. 18-46).

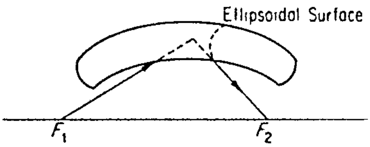


Figure 18-46

EXERCISES 18-9

The following equations define various ellipses. Find the foci, vertices, and sketch the curve.

1. $\frac{x^2}{25} + \frac{y^2}{16} = 1$

2. $25x^2 + 16y^2 = 400$

3. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

4. $\frac{x^2}{18} + \frac{y^2}{50} = 1$

5. $\frac{x^2}{36} + \frac{y^2}{64} = 1$

6. $12x^2 + 15y^2 = 180$

7. $3x^2 + 5y^2 = 15$

8. $12x^2 + 16y^2 = 192$

9. $12x^2 + 12y^2 = 144$

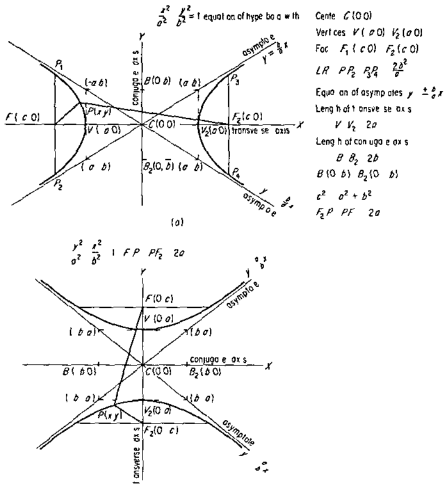
10. $25x^2 + 25y^2 = 1$

18-6b HYPERBOLA: *A hyperbola is the locus of all points at which the difference of the distances from two fixed points is constant.* The two fixed points are called the *foci* and lie on the *transverse axis.* A line perpendicular to the transverse axis, passing through a point midway between the foci, is called the *conjugate axis.* The point of intersection of the axes is called the *center* of the hyperbola. Furthermore, the curve crosses the transverse axis at two points called the *vertices.*

The equations of hyperbolas with centers at the origin and principle axes coincident with coordinate axes along with other properties are listed in Figs 18-47a and 18-47b.

The relationships involving the various properties of the hyperbola are summarized accordingly

$$2c > 2a \text{ and } 2c > 2b\text{; likewise, } c > a \text{ and } c > b.$$



(a)

(b)

Figure 18-47

This indicates that the distance between the foci is greater than the length of either axis Thus $c^2 = a^2 + b^2$ Furthermore, a can be less than equal to, or greater than b This means that the value of a (or b) has no bearing on the position of the transverse axis The positive second degree term will determine the position of the transverse axis and the location of the foci The end points of the conjugate axis $(0, b)$, $(0, -b)$ or $(-b, 0)$, $(b, 0)$ are not on the curve, but are, nevertheless useful quantities in dealing with the hyperbola

The hyperbola has two branches, both opening away from the center These branches approach two lines that pass through the center of the curve called *asymptotes* An *asymptote* is defined as a line approached by a curve as the curve recedes without bound The equations of the asymptotes are included in both Figs 18 47a and 18 47b

In no way are the two branches of the hyperbola to be considered as two parabolas The hyperbola and the parabola are two distinct and separate curves

EXAMPLE 18-S:

The equation of a hyperbola is $144y^2 - 25x^2 = 3{,}600$. Determine the axes, coordinates of foci and vertices, and the end points of the latus rectum and the equations of the asymptotes. Sketch curve.

Solution:

The equation $144y^2 - 25x^2 = 3{,}600$, must be transformed into standard form by dividing through by 3,600.

$$\frac{144y^2}{3{,}600} - \frac{25x^2}{3{,}600} = \frac{3{,}600}{3{,}600}, \text{ which leads to } \frac{y^2}{25} - \frac{x^2}{144} = 1$$

The equation now appears in standard form. By inspection (Fig. 18-47b), it is determined that the hyperbola has its center at the origin (0, 0), its transverse axis on the y-axis, and its conjugate axis on the x-axis.

Furthermore,

$$a^2 = 25 \text{ and } a = \pm 5$$

$$b^2 = 144 \text{ and } b = \pm 12$$

$$c^2 = a^2 + b^2 = 25 + 144 = 169, \text{ and } c = \sqrt{169} = \pm 13$$

With this information, the various properties of the curve can be found.

Foci: $F_1(0, 13)$, $F_2(0, -13)$

Vertices: $V_1(0, 5)$, $V_2(0, -5)$

$B_1(-12, 0)$, $B_2(12, 0)$

Latus Rectum: $LR = \frac{2b^2}{a} = \frac{2(144)}{5} = \frac{288}{5}, \frac{b^2}{a} = \frac{144}{5}$

End Points of LR: $P_1\left(-\frac{144}{5}, 13\right)$, $P_2\left(\frac{144}{5}, 13\right)$

$P_3\left(-\frac{144}{5}, -13\right)$, $P_4\left(\frac{144}{5}, -13\right)$

Equations of asymptotes are $y = \pm\frac{a}{b}x$, or $y = \frac{5}{12}x$ and $y = -\frac{5}{12}x$.

The sketch follows (Fig. 18-48).

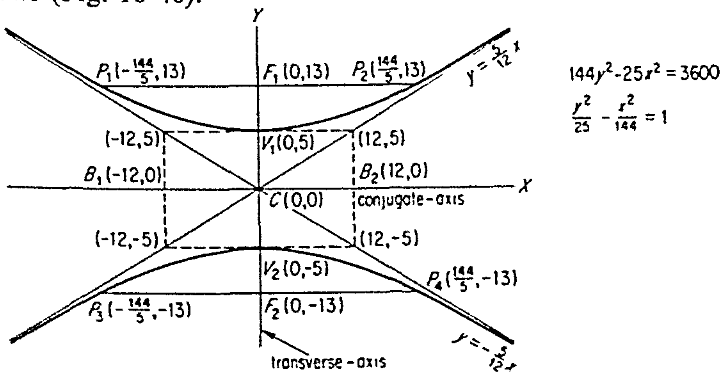


Figure 18-48

The following equations define various hyperbolas Find the foci, vertices, equation of asymptotes, and sketch the curve

1. $16x^2 - 9y^2 = 144$	2. $9y^2 - 16x^2 = 144$
3. $x^2 - 16y^2 = 16$	4. $x^2 - y^2 = 25$
5. $y^2 - 9x^2 = 9$	6 $y^2 - 9x^2 = -9$
7. $16y^2 - 4x^2 = 64$	8 $25y^2 - 4x^2 = 100$

18 6c SOME SPECIAL CASES OF THE CONICS The general appearance of an ellipse (round or flat) is measured by a ratio called *eccentricity*, e where $e = c/a$ Since $a > c$, this ratio will always be less than 1, actually, $0 < e < 1$ As the distance between the foci $2c$, becomes smaller, the ellipse approaches the form of a circle If $c = 0$ then $e = 0$ and the resulting curve is a circle where $a = b = r$ (If $c = 0$ $a^2 = b^2 + c^2$ becomes $a^2 = b^2 + 0$, and $a^2 = b^2$ or $a = b$)

Thus, the special case of an ellipse is a circle, or a circle is an ellipse with eccentricity = 0 Several conditions are sketched, with e taking on several values (Fig 18 49)

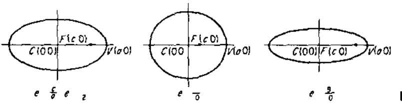


Figure 18-49

The hyperbola becomes two intersecting straight lines passing through the origin (center of the hyperbola) when the equation $x^2/a^2 - y^2/b^2 = 1$ takes on the form $x^2/a^2 - y^2/b^2 = 0$, here $y^2 = b^2/a^2x^2$, or $y = \pm b/ax$, which turn out to be the equations of the asymptotes For example, $x^2/4 - y^2/9 = 0$, leads to the following equations of two intersecting straight lines $y = \pm\frac{3}{2}x$ (Fig 18-50)

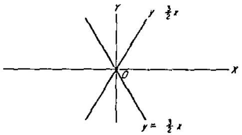


Figure 18-50

The concept of a point as a special case of a conic may be interpreted as a circle with radius equal to zero, or $x^2 + y^2 = 0$, where (0, 0) satisfies the equation

The equation $xy = 1$ defines an *equilateral hyperbola* (Fig 18-51) The

coordinate axes are the asymptotes of the curve, whereas the axes of the equilateral hyperbola are the lines $y = \pm x$.

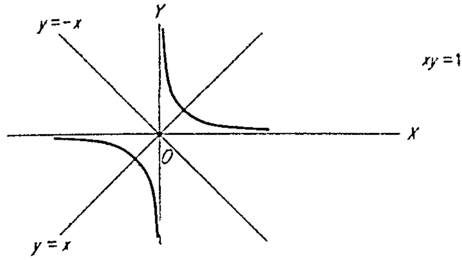


Figure 18-51

REVIEW EXERCISES 18-11

Identify the curves (straight lines) represented by the equations in exercises 1-20. List several properties that will better define the curve.

Example: $x^2 + 2y^2 = 2$; in standard form, the equation takes on the following appearance: $x^2/2 + y^2/1 = 1$. The curve is an ellipse with its center at (0, 0), its major axis on the x-axis, its vertices at $(-\sqrt{2}, 0)$, $(\sqrt{2}, 0)$, and its foci at $(-1, 0)$, $(1, 0)$.

1. $2y^2 = 5x$

2. $5x^2 = 2y$

3. $y^2 = 4x$

4. $3x + 5 = 0$

5. $3x^2 - 2 = 1$

6. $x = y$

7. $3x^2 - 5y^2 = 15$

8. $9x^2 + 9y^2 = 1$

9. $9x^2 - 9y^2 = 1$

10. $9x^2 - 9y^2 = 0$

11. $x^2 + y^2 = 12$

12. $4x^2 - 4x + 1 = 0$

13. $9y^2 + 16x^2 = 144$

14. $16x^2 - 9y^2 = 144$

15. $3y^2 + 7y = 20$

16. $25x^2 + 25y^2 = 625$

17. $y^2 - 24x = 0$

18. $\frac{x^2}{2} - \frac{y^2}{2} = 0$

19. $x^2 - y^2 - 4x - 4y = 1$

20. $-xy = 1$

Appendices

0°

′	Sin	Tan	Ctn	Cos	′
0	00000	00000	∞	1.0000	60
1	00029	00029	3437.7	1.0000	59
2	00058	00058	1718.9	1.0000	58
3	00087	00087	1145.9	1.0000	57
4	00116	00116	859.44	1.0000	56
5	00145	00145	687.55	1.0000	55
6	00175	00175	572.96	1.0000	54
7	00204	00204	491.11	1.0000	53
8	00233	00233	429.72	1.0000	52
9	00262	00262	381.97	1.0000	51
10	00291	00291	343.77	1.0000	50
11	00320	00320	312.52	99999	49
12	00349	00349	286.48	99999	48
13	00378	00378	264.44	99999	47
14	00407	00407	245.55	99999	46
15	00436	00436	229.18	99999	45
16	00465	00465	214.86	99999	44
17	00495	00495	202.22	99999	43
18	00574	00524	190.98	99999	42
19	00553	00553	180.93	99998	41
20	00582	00582	171.89	99998	40
21	00611	00611	163.70	99998	39
22	00640	00640	156.26	99998	38
23	00669	00669	149.47	99998	37
24	00698	00698	143.24	99998	36
25	00727	00727	137.51	99997	35
26	00756	00756	132.22	99997	34
27	00785	00785	127.32	99997	33
28	00814	00815	122.77	99997	32
29	00844	00844	118.54	99996	31
30	00873	00873	114.59	99996	30
31	00902	00902	110.89	99996	29
32	00931	00931	107.43	99996	28
33	00960	00960	104.17	99995	27
34	00989	00989	101.11	99995	26
35	01018	01018	98.218	99995	25
36	01047	01047	95.489	99995	24
37	01076	01076	92.908	99994	23
38	01105	01105	90.463	99994	22
39	01134	01135	88.144	99994	21
40	01164	01164	85.940	99993	20
41	01193	01193	83.844	99993	19
42	01722	01222	81.847	99993	18
43	01251	01251	79.943	99992	17
44	01280	01280	78.126	99992	16
45	01309	01309	76.390	99991	15
46	01338	01338	74.729	99991	14
47	01367	01367	73.139	99991	13
48	01396	01396	71.615	99990	12
49	01425	01425	70.153	99990	11
50	01454	01455	68.750	99989	10
51	01483	01484	67.402	99989	9
52	01513	01513	66.105	99989	8
53	01542	01542	64.858	99988	7
54	01571	01571	63.657	99988	6
55	01600	01600	62.499	99987	5
56	01629	01629	61.383	99987	4
57	01658	01658	60.306	99986	3
58	01687	01687	59.266	99986	2
59	01716	01716	58.261	99985	1
60	01745	01746	57.290	99985	0
′	Cos	Ctn	Tan	Sin	′

89°

1°

′	Sin	Tan	Ctn	Cos	′
0	01745	01746	57.290	99985	60
1	01774	01775	56.351	99984	59
2	01803	01804	55.442	99984	58
3	01832	01833	54.561	99983	57
4	01862	01862	53.709	99983	56
5	01891	01891	52.882	99982	55
6	01920	01920	52.081	99982	54
7	01949	01949	51.303	99981	53
8	01978	01978	50.549	99980	52
9	02007	02007	49.816	99980	51
10	02036	02036	49.104	99979	50
11	02065	02066	48.412	99979	49
12	02094	02095	47.740	99978	48
13	02123	02124	47.085	99977	47
14	02152	02153	46.449	99977	46
15	02181	02182	45.829	99976	45
16	02211	02211	45.226	99976	44
17	02240	02240	44.639	99975	43
18	02269	02269	44.066	99974	42
19	02298	02298	43.508	99974	41
20	02327	02328	42.964	99973	40
21	02356	02357	42.433	99972	39
22	02385	02386	41.916	99972	38
23	02414	02415	41.411	99971	37
24	02443	02444	40.917	99970	36
25	02472	02473	40.436	99969	35
26	02501	02502	39.965	99969	34
27	02530	02531	39.506	99968	33
28	02560	02560	39.057	99967	32
29	02589	02589	38.618	99966	31
30	02618	02619	38.188	99966	30
31	02647	02648	37.769	99965	29
32	02676	02677	37.358	99964	28
33	02705	02706	36.956	99963	27
34	02734	02735	36.563	99963	26
35	02763	02764	36.178	99962	25
36	02792	02793	35.801	99961	24
37	02821	02822	35.431	99960	23
38	02850	02851	35.070	99959	22
39	02879	02881	34.715	99959	21
40	02908	02910	34.368	99958	20
41	02938	02939	34.027	99957	19
42	02967	02968	33.694	99956	18
43	02996	02997	33.366	99955	17
44	03025	03026	33.045	99954	16
45	03054	03055	32.730	99953	15
46	03083	03084	32.421	99952	14
47	03112	03114	32.118	99952	13
48	03141	03143	31.821	99951	12
49	03170	03172	31.528	99950	11
50	03199	03201	31.242	99949	10
51	03228	03230	30.960	99948	9
52	03257	03259	30.683	99947	8
53	03286	03288	30.412	99946	7
54	03316	03317	30.145	99945	6
55	03345	03346	29.882	99944	5
56	03374	03376	29.624	99943	4
57	03403	03405	29.371	99942	3
58	03432	03434	29.122	99941	2
59	03461	03463	28.877	99940	1
60	03490	03492	28.636	99939	0
′	Cos	Ctn	Tan	Sin	′

88°

2°

′	Sin	Tan	Ctn	Cos	′
0	.03490	.03492	28 636	.99939	60
1	.03519	.03521	28 399	.99938	59
2	03548	.03550	28 166	.99937	58
3	.03577	.03579	27 937	.99936	57
4	.03606	.03609	27.712	.99935	56
5	.03635	.03638	27 490	.99934	55
6	.03664	.03667	27.271	.99933	54
7	03693	.03696	27 057	.99932	53
8	.03723	.03725	26 845	99931	52
9	.03752	.03754	26 637	99930	51
10	03781	.03783	26 432	.99929	50
11	.03810	.03812	26 230	.99927	49
12	.03839	.03842	26 031	.99926	48
13	.03868	.03871	25 835	.99925	47
14	.03897	.03900	25 642	.99924	46
15	.03926	.03929	25 452	.99923	45
16	.03955	03958	25 264	.99922	44
17	.03984	03987	25 080	.99921	43
18	.04013	04016	24.898	99919	42
19	.04042	.04046	24 719	.99918	41
20	.04071	.04075	24.542	99917	40
21	.04100	04104	24 368	.99916	39
22	.04129	.04133	24 196	.99915	38
23	.04159	.04162	24 026	.99913	37
24	.04188	.04191	23 859	.99912	36
25	.04217	.04220	23 695	.99911	35
26	.04246	.04250	23 532	.99910	34
27	.04275	.04279	23 372	.99909	33
28	.04304	.04308	23 214	99907	32
29	.04333	.04337	23 058	.99906	31
30	.04362	04366	22 904	.99905	30
31	.04391	.04395	22 752	.99904	29
32	.04420	.04424	22 602	.99902	28
33	.04449	.04454	22 454	.99901	27
34	.04478	.04483	22.308	.99900	26
35	.04507	.04512	22.164	.99898	25
36	.04536	.04541	22 022	.99897	24
37	.04565	.04570	21.881	.99896	23
38	04594	04599	21.743	.99894	22
39	.04623	.04628	21.606	.99893	21
40	.04653	.04658	21.470	.99892	20
41	.04682	.04687	21.337	.99890	19
42	04711	.04716	21 205	.99889	18
43	04740	.04745	21 075	.99888	17
44	04769	04774	20.946	.99886	16
45	.04798	04803	20 819	.99885	15
46	04827	04833	20 693	.99883	14
47	04856	.04862	20 569	.99882	13
48	04885	.04891	20 446	.99881	12
49	.04914	.04920	20 325	.99879	11
50	.04943	.04949	20 206	.99878	10
51	04972	.04978	20 087	.99876	9
52	.05001	.05007	19 970	.99875	8
53	.05030	.05037	19 855	.99873	7
54	.05059	.05066	19 740	.99872	6
55	.05088	.05095	19 627	.99870	5
56	.05117	.05124	19 516	.99869	4
57	.05146	.05153	19 405	.99867	3
58	.05175	.05182	19 296	.99866	2
59	.05205	.05212	19 188	.99864	1
60	.05234	.05241	19 081	.99863	0
′	Cos	Ctn	Tan	Sin	′

87°

3°

′	Sin	Tan	Ctn	Cos	′
0	05234	.05241	19 081	99863	60
1	05263	05270	18 976	99861	59
2	05292	05299	18 871	99860	58
3	05321	05328	18 768	99858	57
4	05350	05357	18 666	99857	56
5	05379	05387	18 564	99855	55
6	05408	05416	18 464	99854	54
7	05437	05445	18 366	99852	53
8	05466	05474	18 268	99851	52
9	05495	05503	18 171	99849	51
10	05524	05533	18 075	99847	50
11	05553	05562	17 980	99846	49
12	05582	05591	17 886	99844	48
13	05611	05620	17 793	99842	47
14	05640	05649	17 702	99841	46
15	05669	05678	17 611	99839	45
16	05698	05708	17 521	99838	44
17	05727	05737	17 431	99836	43
18	05756	05766	17 343	99834	42
19	.05785	05795	17 256	99833	41
20	05814	05824	17 169	99831	40
21	05844	05854	17 084	99829	39
22	05873	05883	16 999	99827	38
23	.05902	05912	16 915	99826	37
24	05931	05941	16 832	99824	36
25	05960	05970	16 750	99822	35
26	05989	05999	16 668	99821	34
27	06018	06029	16 587	99819	33
28	06047	06058	16 507	99817	32
29	06076	06087	16 428	99815	31
30	.06105	.06116	16 350	99813	30
31	06134	06145	16 272	99812	29
32	06163	06175	16 195	.99810	28
33	06192	06204	16 119	99808	27
34	.06221	06233	16 043	99806	26
35	06250	06262	15 969	99804	25
36	.06279	06291	15 895	99803	24
37	06308	06321	15 821	99801	23
38	.06337	06350	15 748	99799	22
39	06366	06379	15 676	99797	21
40	06395	06408	15 605	99795	20
41	06424	06438	15 534	99793	19
42	06453	06467	15 464	99792	18
43	06482	06496	15 394	99790	17
44	06511	06525	15 325	99788	16
45	06540	06554	15 257	99786	15
46	06569	06584	15 189	99784	14
47	06598	06613	15 122	99782	13
48	06627	06642	15 056	99780	12
49	06656	06671	14 990	99778	11
50	06685	06700	14 924	99776	10
51	06714	06730	14 860	99774	9
52	06743	06759	14 795	99772	8
53	06773	06788	14 732	99770	7
54	06802	06817	14 669	99768	6
55	06831	06847	14 606	99766	5
56	06860	06876	14 544	99764	4
57	06889	06905	14 482	99762	3
58	.06918	06934	14 421	99760	2
59	06947	06963	14 361	99758	1
60	.06976	06993	14 301	99756	0
′	Cos	Ctn	Tan	Sin	′

86°

4°

′	Sin	Tan	Ctn	Cos	′
0	06976	06993	14 301	99756	60
1	07005	07022	14 241	99754	59
2	07034	07051	14 182	99752	58
3	07063	07080	14 124	99750	57
4	07092	07110	14 065	99748	56
5	07121	07139	14 008	99746	55
6	07150	07168	13 951	99744	54
7	07179	07197	13 894	99742	53
8	07208	07227	13 838	99740	52
9	07237	07256	13 782	99738	51
10	07266	07285	13 727	99736	50
11	07295	07314	13 672	99734	49
12	07324	07344	13 617	99731	48
13	07353	07373	13 563	99729	47
14	07382	07402	13 510	99727	46
15	07411	07431	13 457	99725	45
16	07440	07461	13 404	99723	44
17	07469	07490	13 352	99721	43
18	07498	07519	13 300	99719	42
19	07527	07548	13 248	99716	41
20	07556	07578	13 197	99714	40
21	07585	07607	13 146	99712	39
22	07614	07636	13 096	99710	38
23	07643	07665	13 046	99708	37
24	07672	07695	12 996	99705	36
25	07701	07724	12 947	99703	35
26	07730	07753	12 898	99701	34
27	07759	07782	12 850	99699	33
28	07788	07812	12 801	99696	32
29	07817	07841	12 754	99694	31
30	07846	07870	12 706	99692	30
31	07875	07899	12 659	99689	29
32	07904	07929	12 612	99687	28
33	07933	07958	12 566	99685	27
34	07962	07987	12 520	99683	26
35	07991	08017	12 474	99680	25
36	08020	08046	12 429	99678	24
37	08049	08075	12 384	99676	23
38	08078	08104	12 339	99673	22
39	08107	08134	12 295	99671	21
40	08136	08163	12 251	99668	20
41	08165	08192	12 207	99666	19
42	08194	08221	12 163	99664	18
43	08223	08251	12 120	99661	17
44	08252	08280	12 077	99659	16
45	08281	08309	12 035	99657	15
46	08310	08339	11 992	99654	14
47	08339	08368	11 950	99652	13
48	08368	08397	11 909	99649	12
49	08397	08427	11 867	99647	11
50	08426	08456	11 826	99644	10
51	08455	08485	11 785	99642	9
52	08484	08514	11 745	99639	8
53	08513	08544	11 705	99637	7
54	08542	08573	11 664	99635	6
55	08571	08602	11 625	99632	5
56	08600	08632	11 585	99630	4
57	08629	08661	11.546	99627	3
58	08658	08690	11 507	99625	2
59	08687	08720	11 468	99622	1
60	08716	08749	11 430	99619	0
′	Cos	Ctn	Tan	Sin	′

85°

5°

′	Sin	Tan	Ctn	Cos	′
0	08716	08749	11 430	99619	60
1	08745	08778	11 392	99617	59
2	08774	08807	11 354	99614	58
3	08803	08837	11 316	99612	57
4	08831	08866	11 279	99609	56
5	08860	08895	11 242	99607	55
6	08889	08925	11 205	99604	54
7	08918	08954	11 168	99602	53
8	08947	08983	11 132	99599	52
9	08976	09013	11 095	99596	51
10	09005	09042	11 059	99594	50
11	09034	00971	11 024	99591	49
12	09063	09101	10 988	99588	48
13	09092	09130	10 953	99586	47
14	09121	09159	10 918	99583	46
15	09150	09189	10 883	99580	45
16	09179	09218	10 848	99578	44
17	09208	09247	10 814	99575	43
18	09237	09277	10 780	99572	42
19	09266	09306	10 746	99570	41
20	09295	09335	10 712	99567	40
21	09324	09365	10 678	99564	39
22	09353	09394	10 645	99562	38
23	09382	09423	10 612	99559	37
24	09411	09453	10 579	99556	36
25	09440	09482	10 546	99553	35
26	09469	09511	10 514	99551	34
27	09498	09541	10 481	99548	33
28	09527	09570	10 449	99545	32
29	09556	09600	10 417	99542	31
30	09585	09629	10 385	99540	30
31	09614	09658	10 354	99537	29
32	09642	09688	10 322	99534	28
33	09671	09717	10 291	99531	27
34	09700	09746	10 260	99528	26
35	09729	09776	10 229	99526	25
36	09758	09805	10 199	99523	24
37	09787	09834	10 168	99520	23
38	09816	09864	10 138	99517	22
39	09845	09893	10 108	99514	21
40	09874	09923	10 078	99511	20
41	09903	09952	10 048	99508	19
42	09932	09981	10 019	99506	18
43	09961	10011	9 9893	99503	17
44	09990	10040	9 9601	99500	16
45	10019	10069	9 9310	99497	15
46	10048	10099	9 9021	99494	14
47	10077	10128	9 8734	99491	13
48	10106	10158	9 8448	99488	12
49	10135	10187	9 8164	99485	11
50	10164	10216	9 7882	99482	10
51	10192	10246	9 7601	99479	9
52	10221	10275	9 7322	99476	8
53	10250	10305	9 7044	99473	7
54	10279	10334	9 6768	99470	6
55	10308	10363	9 6493	99467	5
56	10337	10393	9 6220	99464	4
57	10366	10422	9 5949	99461	3
58	10395	10452	9 5679	99458	2
59	10424	10481	9 5411	99455	1
60	10453	10510	9 5144	99452	0
′	Cos	Ctn	Tan	Sin	′

84°

6°

′	Sin	Tan	Ctn	Cos	′
0	.10453	.10510	9 5144	.99452	60
1	10482	.10540	9 4878	.99449	59
2	.10511	.10569	9 4614	.99446	58
3	.10540	.10599	9 4352	99443	57
4	.10569	.10628	9 4090	.99440	56
5	.10597	.10657	9 3831	.99437	55
6	.10626	.10687	9 3572	.99434	54
7	.10655	.10716	9 3315	99431	53
8	.10684	.10746	9 3060	.99428	52
9	.10713	.10775	9 2806	.99424	51
10	.10742	10805	9 2553	.99421	50
11	.10771	.10834	9 2302	99418	49
12	10800	.10863	9 2052	.99415	48
13	.10829	.10893	9 1803	.99412	47
14	.10858	.10922	9 1555	.99409	46
15	.10887	.10952	9 1309	99406	45
16	.10916	.10981	9 1065	.99402	44
17	.10945	.11011	9 0821	.99399	43
18	.10973	.11040	9 0579	.99396	42
19	.11002	.11070	9 0338	.99393	41
20	.11031	.11099	9 0098	.99390	40
21	.11060	.11128	8 9860	.99386	39
22	.11089	.11158	8 9623	.99383	38
23	.11118	.11187	8 9387	.99380	37
24	11147	.11217	8 9152	.99377	36
25	.11176	.11246	8 8919	.99374	35
26	11205	.11276	8 8686	.99370	34
27	11234	.11305	8 8455	99367	33
28	.11263	.11335	8 8225	99364	32
29	.11291	.11364	8 7996	99360	31
30	11320	.11394	8 7769	.99357	30
31	.11349	.11423	8 7542	99354	29
32	.11378	.11452	8 7317	.99351	28
33	11407	.11482	8 7093	.99347	27
34	11436	.11511	8 6870	.99344	26
35	.11465	11541	8 6648	.99341	25
36	11494	.11570	8 6427	99337	24
37	.11523	.11600	8 6208	.99334	23
38	.11552	.11629	8 5989	99331	22
39	.11580	11659	8.5772	.99327	21
40	.11609	.11688	8 5555	.99324	20
41	11638	.11718	8 5340	.99320	19
42	.11667	.11747	8 5126	.99317	18
43	11696	.11777	8 4913	99314	17
44	.11725	11806	8 4701	.99310	16
45	.11754	.11836	8 4490	.99307	15
46	.11783	.11865	8 4280	99303	14
47	11812	.11895	8 4071	.99300	13
48	.11840	.11924	8 3863	.99297	12
49	.11869	.11954	8 3656	.99293	11
50	.11898	.11983	8 3450	99290	10
51	.11927	.12013	8 3245	.99286	9
52	11956	.12042	8 3041	.99283	8
53	.11985	.12072	8 2838	.99279	7
54	.12014	.12101	8 2636	.99276	6
55	.12043	.12131	8 2434	.99272	5
56	.12071	.12160	8 2234	.99269	4
57	12100	.12190	8 2035	.99265	3
58	.12129	.12219	8 1837	.99262	2
59	.12158	.12249	8 1640	.99258	1
60	.12187	.12278	8 1443	.99255	0
′	Cos	Ctn	Tan	Sin	′

83°

7°

′	Sin	Tan	Ctn	Cos	′
0	.12187	12278	8 1443	99255	60
1	12216	.12308	8 1248	99251	59
2	.12245	12338	8 1054	99248	58
3	.12274	12367	8 0860	99244	57
4	.12302	12397	8 0667	99240	56
5	.12331	12426	8 0476	99237	55
6	12360	12456	8 0285	99233	54
7	.12389	12485	8 0095	99230	53
8	.12418	12515	7 9906	99226	52
9	.12447	12544	7 9718	99222	51
10	12476	12574	7 9530	99219	50
11	12504	12603	7 9344	99215	49
12	.12533	12633	7 9158	99211	48
13	12562	12662	7 8973	99208	47
14	12591	12692	7 8789	99204	46
15	12620	.12722	7 8606	99200	45
16	.12649	12751	7 8424	99197	44
17	12678	12781	7 8243	99193	43
18	12706	12810	7 8062	99189	42
19	12735	12840	7 7882	99186	41
20	12764	12869	7 7704	99182	40
21	12793	12899	7 7525	99178	39
22	12822	12929	7 7348	99175	38
23	12851	12958	7 7171	99171	37
24	12880	12988	7 6996	99167	36
25	12908	13017	7 6821	99163	35
26	12937	13047	7 6647	99160	34
27	12966	13076	7 6473	99156	33
28	12995	13106	7 6301	99152	32
29	13024	13136	7 6129	99148	31
30	13053	13165	7 5958	99144	30
31	13081	13195	7 5787	99141	29
32	13110	13224	7 5618	99137	28
33	13139	13254	7 5449	99133	27
34	13168	13284	7 5281	99129	26
35	13197	13313	7 5113	99125	25
36	13226	13343	7 4947	99122	24
37	13254	13372	7 4781	99118	23
38	13283	13402	7 4615	99114	22
39	13312	13432	7 4451	99110	21
40	13341	13461	7 4287	99106	20
41	13370	13491	7 4124	99102	19
42	.13399	13521	7 3962	99098	18
43	13427	13550	7 3800	99094	17
44	13456	13580	7 3639	99091	16
45	13485	13609	7 3479	99087	15
46	13514	13639	7 3319	99083	14
47	13543	13669	7 3160	99079	13
48	13572	13698	7 3002	99075	12
49	13600	13728	7 2844	99071	11
50	13629	.13758	7 2687	99057	10
51	.13658	13787	7 2531	99063	9
52	13687	13817	7 2375	99059	8
53	13716	13846	7 2220	99055	7
54	.13744	13876	7 2066	99051	6
55	.13773	13906	7 1912	99047	5
56	13802	13935	1759	99043	4
57	13831	13965	7 1607	99039	3
58	.13860	13995	7 1455	99035	2
59	13889	14024	7 1304	99031	1
60	13917	14054	7 1154	93027	0
′	Cos	Ctn	Tan	Sin	′

82°

8°

′	Sin	Tan	Ctn	Cos	′
0	13917	14054	7 1154	99027	60
1	13946	14084	7 1004	99023	59
2	13975	14113	7 0855	99019	58
3	14004	14143	7 0706	99015	57
4	14033	14173	7 0558	99011	56
5	14061	14202	7 0410	99006	55
6	14090	14232	7 0264	99002	54
7	14119	14262	7 0117	98998	53
8	14148	14291	6 9972	98994	52
9	14177	14321	6 9827	98990	51
10	14205	14351	6 9682	98986	50
11	14234	14381	6 9538	98982	49
12	14263	14410	6 9395	98978	48
13	14292	14440	6 9252	98973	47
14	14320	14470	6 9110	98969	46
15	14349	14499	6 8969	98965	45
16	14378	14529	6 8828	98961	44
17	14407	14559	6 8687	98957	43
18	14436	14588	6 8548	98953	42
19	14464	14618	6 8408	98948	41
20	14493	14648	6 8269	98944	40
21	14522	14678	6 8131	98940	39
22	14551	14707	6 7994	98936	38
23	14580	14737	6 7856	98931	37
24	14608	14767	6 7720	98927	36
25	14637	14796	6 7584	98923	35
26	14666	14826	6 7448	98919	34
27	14695	14856	6 7313	98914	33
28	14723	14886	6 7179	98910	32
29	14752	14915	6 7045	98906	31
30	14781	14945	6 6912	98902	30
31	14810	14975	6 6779	98897	29
32	14838	15005	6 6646	98893	28
33	14867	15034	6 6514	98889	27
34	14896	15064	6 6383	98884	26
35	14925	15094	6 6252	98880	25
36	14954	15124	6 6122	98876	24
37	14982	15153	6 5992	98871	23
38	15011	15183	6 5863	98867	22
39	15040	15213	6 5734	98863	21
40	15069	15243	6 5606	98858	20
41	15097	15272	6 5478	98854	19
42	15126	15302	6 5350	98849	18
43	15155	15332	6 5223	98845	17
44	15184	15362	6 5097	98841	16
45	15212	15391	6 4971	98836	15
46	15241	15421	6 4846	98832	14
47	15270	15451	6 4721	98827	13
48	15299	15481	6 4596	98823	12
49	15327	15511	6 4472	98818	11
50	15356	15540	6 4348	98814	10
51	15385	15570	6 4225	98809	9
52	15414	15600	6 4103	98805	8
53	15442	15630	6 3980	98800	7
54	15471	15660	6 3859	98796	6
55	15500	15689	6 3737	98791	5
56	15529	15719	6 3617	98787	4
57	15557	15749	6 3496	98782	3
58	15586	15779	6 3376	98778	2
59	15615	15809	6 3257	98773	1
60	15643	15838	6 3138	98769	0
′	Cos	Ctn	Tan	Sin	′

81°

9°

′	Sin	Tan	Ctn	Cos	′
0	15643	15838	6 3138	98769	60
1	15672	15868	6 3019	98764	59
2	15701	15898	6 2901	98760	58
3	15730	15928	6 2783	98755	57
4	15758	15958	6 2666	98751	56
5	15787	15988	6 2549	98746	55
6	15816	16017	6 2432	98741	54
7	15845	16047	6 2316	98737	53
8	15873	16077	6 2200	98732	52
9	15902	16107	6 2085	98728	51
10	15931	16137	6 1970	98723	50
11	15959	16167	6 1856	98718	49
12	15988	16196	6 1742	98714	48
13	16017	16226	6 1628	98709	47
14	16046	16256	6 1515	98704	46
15	16074	16286	6 1402	98700	45
16	16103	16316	6 1290	98695	44
17	16132	16346	6 1178	98690	43
18	16160	16376	6 1066	98686	42
19	16189	16405	6 0955	98681	41
20	16218	16435	6 0844	98676	40
21	16246	16465	6 0734	98671	39
22	16275	16495	6 0624	98667	38
23	16304	16525	6 0514	98662	37
24	16333	16555	6 0405	98657	36
25	16361	16585	6 0296	98652	35
26	16390	16615	6 0188	98648	34
27	16419	16645	6 0080	98643	33
28	16447	16674	5 9972	98638	32
29	16476	16704	5 9865	98633	31
30	16505	16734	5 9758	98629	30
31	16533	16764	5 9651	98624	29
32	16562	16794	5 9545	98619	28
33	16591	16824	5 9439	98614	27
34	16620	16854	5 9333	98609	26
35	16648	16884	5 9228	98604	25
36	16677	16914	5 9124	98600	24
37	16706	16944	5 9019	98595	23
38	16734	16974	5 8915	98590	22
39	16763	17004	5 8811	98585	21
40	16792	17033	5 8708	98580	20
41	16820	17063	5 8605	98575	19
42	16849	17093	5 8502	98570	18
43	16878	17123	5 8400	98565	17
44	16906	17153	5 8298	98561	16
45	16935	17183	5 8197	98556	15
46	16964	17213	5 8095	98551	14
47	16992	17243	5 7994	98546	13
48	17021	17273	5 7894	98541	12
49	17050	17303	5 7794	98536	11
50	17078	17333	5 7694	98531	10
51	17107	17363	5 7594	98526	9
52	17136	17393	5 7495	98521	8
53	17164	17423	5 7396	98516	7
54	17193	17453	5 7297	98511	6
55	17222	17483	5 7199	98506	5
56	17250	17513	5 7101	98501	4
57	17279	17543	5 7004	98496	3
58	17308	17573	5 6906	98491	2
59	17336	17603	5 6809	98486	1
60	17365	17633	5 6713	98481	0
′	Cos	Ctn	Tan	Sin	′

80°

10°

′	Sin	Tan	Ctn	Cos	′
0	17365	17633	5 6713	98481	60
1	17393	17663	5 6617	98476	59
2	17422	17693	5 6521	98471	58
3	17451	17723	5 6425	98466	57
4	17479	17753	5 6329	98461	56
5	17508	17783	5 6234	98455	55
6	17537	17813	5 6140	98450	54
7	17565	17843	5 6045	98445	53
8	17594	17873	5 5951	98440	52
9	17623	17903	5 5857	98435	51
10	17651	17933	5 5764	98430	50
11	17680	17963	5 5671	98425	49
12	17708	17993	5 5578	98420	48
13	17737	18023	5 5485	98414	47
14	17766	18053	5 5393	98409	46
15	17794	18083	5 5301	98404	45
16	17823	18113	5 5209	98399	44
17	17852	18143	5 5118	98394	43
18	17880	18173	5 5026	98389	42
19	17909	18203	5 4936	98383	41
20	17937	18233	5 4845	98378	40
21	17966	18263	5 4755	98373	39
22	17995	18293	5 4665	98368	38
23	18023	18323	5 4575	98362	37
24	18052	18353	5 4486	98357	36
25	18081	18384	5 4397	98352	35
26	18109	18414	5 4308	98347	34
27	18138	18444	5 4219	98341	33
28	18166	18474	5 4131	98336	32
29	18195	18504	5 4043	98331	31
30	18224	18534	5 3955	98325	30
31	18252	18564	5 3868	98320	29
32	18281	18594	5 3781	98315	28
33	18309	18624	5 3694	98310	27
34	18338	18654	5 3607	98304	26
35	18367	18684	5 3521	98299	25
36	18395	18714	5 3435	98294	24
37	18424	18745	5 3349	98288	23
38	18452	18775	5 3263	98283	22
39	18481	18805	5 3178	98277	21
40	18509	18835	5 3093	98272	20
41	18538	18865	5 3008	98267	19
42	18567	18895	5 2924	98261	18
43	18595	18925	5 2839	98256	17
44	18624	18955	5 2755	98250	16
45	18652	18986	5 2672	98245	15
46	18681	19016	5 2588	98240	14
47	18710	19046	5 2505	98234	13
48	18738	19076	5 2422	98229	12
49	18767	19106	5 2339	98223	11
50	18795	19136	5 2257	98218	10
51	18824	19166	5 2174	98212	9
52	18852	19197	5 2092	98207	8
53	18881	19227	5 2011	98201	7
54	18910	19257	5 1929	98196	6
55	18938	19287	5 1848	98190	5
56	18967	19317	5 1767	98185	4
57	18995	19347	5 1686	98179	3
58	19024	19378	5 1606	98174	2
59	19052	19408	5 1526	98168	1
60	19081	19438	5 1446	98163	0
′	Cos	Ctn	Tan	Sin	′

79°

11°

′	Sin	Tan	Ctn	Cos	′
0	19081	19438	5 1446	98163	60
1	19109	19468	5 1366	98157	59
2	19138	19498	5 1286	98152	58
3	19167	19529	5 1207	98146	57
4	19195	19559	5 1128	98140	56
5	19224	19589	5 1049	98135	55
6	19252	19619	5 0970	98129	54
7	19281	19649	5 0892	98124	53
8	19309	19680	5 0814	98118	52
9	19338	19710	5 0736	98112	51
10	19366	19740	5 0658	98107	50
11	19395	19770	5 0581	98101	49
12	19423	19801	5 0504	98096	48
13	19452	19831	5 0427	98090	47
14	19481	19861	5 0350	98084	46
15	19509	19891	5 0273	98079	45
16	19538	19921	5 0197	98073	44
17	19566	19952	5 0121	98067	43
18	19595	19982	5 0045	98061	42
19	19623	20012	4 9969	98056	41
20	19652	20042	4 9894	98050	40
21	19680	20073	4 9819	98044	39
22	19709	20103	4 9744	98039	38
23	19737	20133	4 9669	98033	37
24	19766	20164	4 9594	98027	36
25	19794	20194	4 9520	98021	35
26	19823	20224	4 9446	98016	34
27	19851	20254	4 9372	98010	33
28	19880	20285	4 9298	98004	32
29	19908	20315	4 9225	97998	31
30	19937	20345	4 9152	97992	30
31	19965	20376	4 9078	97987	29
32	19994	20406	4 9006	97981	28
33	20022	20436	4 8933	97975	27
34	20051	20466	4 8860	97969	26
35	20079	20497	4 8788	97963	25
36	20108	20527	4 8716	97958	24
37	20136	20557	4 8644	97952	23
38	20165	20588	4 8573	97946	22
39	20193	20618	4 8501	97940	21
40	20222	20648	4 8430	97934	20
41	20250	20679	4 8359	97928	19
42	20279	20709	4 8288	97922	18
43	20307	20739	4 8218	97916	17
44	20336	20770	4 8147	97910	16
45	20364	20800	4 8077	97905	15
46	20393	20830	4 8007	97899	14
47	20421	20861	4 7937	97893	13
48	20450	20891	4 7867	97887	12
49	20478	20921	4 7798	97881	11
50	20507	20952	4 7729	97875	10
51	20535	20982	4 7659	97869	9
52	20563	21013	4 7591	97863	8
53	20592	21043	4 7522	97857	7
54	20620	21073	4 7453	97851	6
55	20649	21104	4 7385	97845	5
56	20677	21134	4 7317	97839	4
57	20706	21164	4 7249	97833	3
58	20734	21195	4 7181	97827	2
59	20763	21225	4 7114	97821	1
60	20791	21256	4 7046	97815	0
′	Cos	Ctn	Tan	Sin	′

78°

12°

′	Sin	Tan	Ctn	Cos	′
0	20791	21256	4.7046	97815	60
1	20820	21286	4.6979	97809	59
2	20848	21316	4.6912	97803	58
3	20877	21347	4.6845	97797	57
4	20905	21377	4.6779	97791	56
5	20933	21408	4.6712	97784	55
6	20962	21438	4.6646	97778	54
7	20990	21469	4.6580	97772	53
8	21019	21499	4.6514	97766	52
9	21047	21529	4.6448	97760	51
10	21076	21560	4.6382	97754	50
11	21104	21590	4.6317	97748	49
12	21132	21621	4.6252	97742	48
13	21161	21651	4.6187	97735	47
14	21189	21682	4.6122	97729	46
15	21218	21712	4.6057	97723	45
16	21246	21743	4.5993	97717	44
17	21275	21773	4.5928	97711	43
18	21303	21804	4.5864	97705	42
19	21331	21834	4.5800	97698	41
20	21360	21864	4.5736	97692	40
21	21388	21895	4.5673	97686	39
22	21417	21925	4.5609	97680	38
23	21445	21956	4.5546	97673	37
24	21474	21986	4.5483	97667	36
25	21502	22017	4.5420	97661	35
26	21530	22047	4.5357	97655	34
27	21559	22078	4.5294	97648	33
28	21587	22108	4.5232	97642	32
29	21616	22139	4.5169	97636	31
30	21644	22169	4.5107	97630	30
31	21672	22200	4.5045	97623	29
32	21701	22231	4.4983	97617	28
33	21729	22261	4.4922	97611	27
34	21758	22292	4.4860	97604	26
35	21786	22322	4.4799	97598	25
36	21814	22353	4.4737	97592	24
37	21843	22383	4.4676	97585	23
38	21871	22414	4.4615	97579	22
39	21899	22444	4.4555	97573	21
40	21928	22475	4.4494	97566	20
41	21956	22505	4.4434	97560	19
42	21985	22536	4.4373	97553	18
43	22013	22567	4.4313	97547	17
44	22041	22597	4.4253	97541	16
45	22070	22628	4.4194	97534	15
46	22098	22658	4.4134	97528	14
47	22126	22689	4.4075	97521	13
48	22155	22719	4.4015	97515	12
49	22183	22750	4.3956	97508	11
50	22212	22781	4.3897	97502	10
51	22240	22811	4.3838	97496	9
52	22268	22842	4.3779	97489	8
53	22297	22872	4.3721	97483	7
54	22325	22903	4.3662	97476	6
55	22353	22934	4.3604	97470	5
56	22382	22964	4.3546	97463	4
57	22410	22995	4.3488	97457	3
58	22438	23026	4.3430	97450	2
59	22467	23056	4.3372	97444	1
60	22495	23087	4.3315	97437	0
′	Cos	Ctn	Tan	Sin	′

77°

13°

′	Sin	Tan	Ctn	Cos	′
0	22495	23087	4.3315	97437	60
1	22523	23117	4.3257	97430	59
2	22552	23148	4.3200	97424	58
3	22580	23179	4.3143	97417	57
4	22608	23209	4.3086	97411	56
5	22637	23240	4.3029	97404	55
6	22665	23271	4.2972	97398	54
7	22693	23301	4.2916	97391	53
8	22722	23332	4.2859	97384	52
9	22750	23363	4.2803	97378	51
10	22778	23393	4.2747	97371	50
11	22807	23424	4.2691	97365	49
12	22835	23455	4.2635	97358	48
13	22863	23485	4.2580	97351	47
14	22892	23516	4.2524	97345	46
15	22920	23547	4.2468	97338	45
16	22948	23578	4.2413	97331	44
17	22977	23608	4.2358	97325	43
18	23005	23639	4.2303	97318	42
19	23033	23670	4.2248	97311	41
20	23062	23700	4.2193	97304	40
21	23090	23731	4.2139	97298	39
22	23118	23762	4.2084	97291	38
23	23146	23793	4.2030	97284	37
24	23175	23823	4.1976	97278	36
25	23203	23854	4.1922	97271	35
26	23231	23885	4.1868	97264	34
27	23260	23916	4.1814	97257	33
28	23288	23946	4.1760	97251	32
29	23316	23977	4.1706	97244	31
30	23345	24008	4.1653	97237	30
31	23373	24039	4.1600	97230	29
32	23401	24069	4.1547	97223	28
33	23429	24100	4.1493	97217	27
34	23458	24131	4.1441	97210	26
35	23486	24162	4.1388	97203	25
36	23514	24193	4.1335	97196	24
37	23542	24223	4.1282	97189	23
38	23571	24254	4.1230	97182	22
39	23599	24285	4.1178	97176	21
40	23627	24316	4.1126	97169	20
41	23656	24347	4.1074	97162	19
42	23684	24377	4.1022	97155	18
43	23712	24408	4.0970	97148	17
44	23740	24439	4.0918	97141	16
45	23769	24470	4.0867	97134	15
46	23797	24501	4.0815	97127	14
47	23825	24532	4.0764	97120	13
48	23853	24562	4.0713	97113	12
49	23882	24593	4.0662	97106	11
50	23910	24624	4.0611	97100	10
51	23938	24655	4.0560	97093	9
52	23966	24686	4.0509	97086	8
53	23995	24717	4.0459	97079	7
54	24023	24747	4.0408	97072	6
55	24051	24778	4.0358	97065	5
56	24079	24809	4.0308	97058	4
57	24108	24840	4.0257	97051	3
58	24136	24871	4.0207	97044	2
59	24164	24902	4.0158	97037	1
60	24192	24933	4.0108	97030	0
′	Cos	Ctn	Tan	Sin	′

76°

14°

′	Sin	Tan	Ctn	Cos	′
0	.24192	.24933	4.0108	.97030	60
1	.24220	.24964	4.0058	.97023	59
2	.24249	.24995	4.0009	.97015	58
3	.24277	.25026	3.9959	.97008	57
4	.24305	.25056	3.9910	.97001	56
5	.24333	.25087	3.9861	.96994	55
6	.24362	.25118	3.9812	.96987	54
7	.24390	.25149	3.9763	.96980	53
8	.24418	.25180	3.9714	.96973	52
9	.24446	.25211	3.9665	.96966	51
10	.24474	.25242	3.9617	.96959	50
11	.24503	.25273	3.9568	.96952	49
12	.24531	.25304	3.9520	.96945	48
13	.24559	.25335	3.9471	.96937	47
14	.24587	.25366	3.9423	.96930	46
15	.24615	.25397	3.9375	.96923	45
16	.24644	.25428	3.9327	.96916	44
17	.24672	.25459	3.9279	.96909	43
18	.24700	.25490	3.9232	.96902	42
19	.24728	.25521	3.9184	.96894	41
20	.24756	.25552	3.9136	.96887	40
21	.24784	.25583	3.9089	.96880	39
22	.24813	.25614	3.9042	.96873	38
23	.24841	.25645	3.8995	.96866	37
24	.24869	.25676	3.8947	.96858	36
25	.24897	.25707	3.8900	.96851	35
26	.24925	.25738	3.8854	.96844	34
27	.24954	.25769	3.8807	.96837	33
28	.24982	.25800	3.8760	.96829	32
29	.25010	.25831	3.8714	.96822	31
30	.25038	.25862	3.8667	.96815	30
31	.25066	.25893	3.8621	.96807	29
32	.25094	.25924	3.8575	.96800	28
33	.25122	.25955	3.8528	.96793	27
34	.25151	.25986	3.8482	.96786	26
35	.25179	.26017	3.8436	.96778	25
36	.25207	.26048	3.8391	.96771	24
37	.25235	.26079	3.8345	.96764	23
38	.25263	.26110	3.8299	.96756	22
39	.25291	.26141	3.8254	.96749	21
40	.25320	.26172	3.8208	.96742	20
41	.25348	.26203	3.8163	.96734	19
42	.25376	.26235	3.8118	.96727	18
43	.25404	.26266	3.8073	.96719	17
44	.25432	.26297	3.8028	.96712	16
45	.25460	.26328	3.7983	.96705	15
46	.25488	.26359	3.7938	.96697	14
47	.25516	.26390	3.7893	.96690	13
48	.25545	.26421	3.7848	.96682	12
49	.25573	.26452	3.7804	.96675	11
50	.25601	.26483	3.7760	.96667	10
51	.25629	.26515	3.7715	.96660	9
52	.25657	.26546	3.7671	.96653	8
53	.25685	.26577	3.7627	.96645	7
54	.25713	.26608	3.7583	.96638	6
55	.25741	.26639	3.7539	.96630	5
56	.25769	.26670	3.7495	.96623	4
57	.25798	.26701	3.7451	.96615	3
58	.25826	.26733	3.7403	.96608	2
59	.25854	.26764	3.7364	.96600	1
60	.25882	.26795	3.7321	.96593	0
′	Cos	Ctn	Tan	Sin	′

75°

15°

′	Sin	Tan	Ctn	Cos	′
0	.25882	.26795	3.7321	.96593	60
1	.25910	.26826	3.7277	.96585	59
2	.25938	.26857	3.7234	.96578	58
3	.25966	.26888	3.7191	.96570	57
4	.25994	.26920	3.7148	.96562	56
5	.26022	.26951	3.7105	.96555	55
6	.26050	.26982	3.7062	.96547	54
7	.26079	.27013	3.7019	.96540	53
8	.26107	.27044	3.6976	.96532	52
9	.26135	.27076	3.6933	.96524	51
10	.26163	.27107	3.6891	.96517	50
11	.26191	.27138	3.6848	.96509	49
12	.26219	.27169	3.6806	.96502	48
13	.26247	.27201	3.6764	.96494	47
14	.26275	.27232	3.6722	.96486	46
15	.26303	.27263	3.6680	.96479	45
16	.26331	.27294	3.6638	.96471	44
17	.26359	.27326	3.6596	.96463	43
18	.26387	.27357	3.6554	.96456	42
19	.26415	.27388	3.6512	.96448	41
20	.26443	.27419	3.6470	.96440	40
21	.26471	.27451	3.6429	.96433	39
22	.26500	.27482	3.6387	.96425	38
23	.26528	.27513	3.6346	.96417	37
24	.26556	.27545	3.6305	.96410	36
25	.26584	.27576	3.6264	.96402	35
26	.26612	.27607	3.6222	.96394	34
27	.26640	.27638	3.6181	.96386	33
28	.26668	.27670	3.6140	.96379	32
29	.26696	.27701	3.6100	.96371	31
30	.26724	.27732	3.6059	.96363	30
31	.26752	.27764	3.6018	.96355	29
32	.26780	.27795	3.5978	.96347	28
33	.26808	.27826	3.5937	.96340	27
34	.26836	.27858	3.5897	.96332	26
35	.26864	.27889	3.5856	.96324	25
36	.26892	.27921	3.5816	.96316	24
37	.26920	.27952	3.5776	.96308	23
38	.26948	.27983	3.5736	.96301	22
39	.26976	.28015	3.5696	.96293	21
40	.27004	.28046	3.5656	.96285	20
41	.27032	.28077	3.5616	.96277	19
42	.27060	.28109	3.5576	.96269	18
43	.27088	.28140	3.5536	.96261	17
44	.27116	.28172	3.5497	.96253	16
45	.27144	.28203	3.5457	.96246	15
46	.27172	.28234	3.5418	.96238	14
47	.27200	.28266	3.5379	.96230	13
48	.27228	.28297	3.5339	.96222	12
49	.27256	.28329	3.5300	.96214	11
50	.27284	.28360	3.5261	.96206	10
51	.27312	.28391	3.5222	.96198	9
52	.27340	.28423	3.5183	.96190	8
53	.27368	.28454	3.5144	.96182	7
54	.27396	.28486	3.5105	.96174	6
55	.27424	.28517	3.5067	.96166	5
56	.27452	.28549	3.5028	.96158	4
57	.27480	.28580	3.4989	.96150	3
58	.27508	.28612	3.4951	.96142	2
59	.27536	.28643	3.4912	.96134	1
60	.27564	.28675	3.4874	.96126	0
′	Cos	Ctn	Tan	Sin	′

74°

16°

′	Sin	Tan	Ctn	Cos	′
0	27564	28675	3.4874	96126	60
1	27592	28706	3.4836	96118	59
2	27620	28738	3.4798	96110	58
3	27648	28769	3.4760	96102	57
4	27676	28801	3.4722	96094	56
5	27704	28832	3.4684	96086	55
6	27731	28864	3.4646	96078	54
7	27759	28895	3.4608	96070	53
8	27787	28927	3.4570	96062	52
9	27815	28958	3.4533	96054	51
10	27843	28990	3.4495	96046	50
11	27871	29021	3.4458	96037	49
12	27899	29053	3.4420	96029	48
13	27927	29084	3.4383	96021	47
14	27955	29116	3.4346	96013	46
15	27983	29147	3.4308	96005	45
16	28011	29179	3.4271	95997	44
17	28039	29210	3.4234	95989	43
18	28067	29242	3.4197	95981	42
19	28095	29274	3.4160	95972	41
20	28123	29305	3.4124	95964	40
21	28150	29337	3.4087	95956	39
22	28178	29368	3.4050	95948	38
23	28206	29400	3.4014	95940	37
24	28234	29432	3.3977	95931	36
25	28262	29463	3.3941	95923	35
26	28290	29495	3.3904	95915	34
27	28318	29526	3.3868	95907	33
28	28346	29558	3.3832	95898	32
29	28374	29590	3.3796	95890	31
30	28402	29621	3.3759	95882	30
31	28429	29653	3.3723	95874	29
32	28457	29685	3.3687	95865	28
33	28485	29716	3.3652	95857	27
34	28513	29748	3.3616	95849	26
35	28541	29780	3.3580	95841	25
36	28569	29811	3.3544	95832	24
37	28597	29843	3.3509	95824	23
38	28625	29875	3.3473	95816	22
39	28652	29906	3.3438	95807	21
40	28680	29938	3.3402	95799	20
41	28708	29970	3.3367	95791	19
42	28736	30001	3.3332	95782	18
43	28764	30033	3.3297	95774	17
44	28792	30065	3.3261	95766	16
45	28820	30097	3.3226	95757	15
46	28847	30128	3.3191	95749	14
47	28875	30160	3.3156	95740	13
48	28903	30192	3.3122	95732	12
49	28931	30224	3.3087	95724	11
50	28959	30255	3.3052	95715	10
51	28987	30287	3.3017	95707	9
52	29015	30319	3.2983	95698	8
53	29042	30351	3.2948	95690	7
54	29070	30382	3.2914	95681	6
55	29098	30414	3.2879	95673	5
56	29126	30446	3.2845	95664	4
57	29154	30478	3.2811	95656	3
58	29182	30509	3.2777	95647	2
59	29209	30541	3.2743	95639	1
60	29237	30573	3.2709	95630	0
′	Cos	Ctn	Tan	Sin	′

73°

17°

′	Sin	Tan	Ctn	Cos	′
0	29237	30573	3.2709	95630	60
1	29265	30605	3.2675	95622	59
2	29293	30637	3.2641	95613	58
3	29321	30669	3.2607	95605	57
4	29348	30700	3.2573	95596	56
5	29376	30732	3.2539	95588	55
6	29404	30764	3.2506	95579	54
7	29432	30796	3.2472	95571	53
8	29460	30828	3.2438	95562	52
9	29487	30860	3.2405	95554	51
10	29515	30891	3.2371	95545	50
11	29543	30923	3.2338	95536	49
12	29571	30955	3.2305	95528	48
13	29599	30987	3.2272	95519	47
14	29626	31019	3.2238	95511	46
15	29654	31051	3.2205	95502	45
16	29682	31083	3.2172	95493	44
17	29710	31115	3.2139	95485	43
18	29737	31147	3.2106	95476	42
19	29765	31178	3.2073	95467	41
20	29793	31210	3.2041	95459	40
21	29821	31242	3.2008	95450	39
22	29849	31274	3.1975	95441	38
23	29876	31306	3.1943	95433	37
24	29904	31338	3.1910	95424	36
25	29932	31370	3.1878	95415	35
26	29960	31402	3.1845	95407	34
27	29987	31434	3.1813	95398	33
28	30015	31466	3.1780	95389	32
29	30043	31498	3.1748	95380	31
30	30071	31530	3.1716	95372	30
31	30098	31562	3.1684	95363	29
32	30126	31594	3.1652	95354	28
33	30154	31626	3.1620	95345	27
34	30182	31658	3.1588	95337	26
35	30209	31690	3.1556	95328	25
36	30237	31722	3.1524	95319	24
37	30265	31754	3.1492	95310	23
38	30292	31786	3.1450	95301	22
39	30320	31818	3.1429	95293	21
40	30348	31850	3.1397	95284	20
41	30376	31882	3.1366	95275	19
42	30403	31914	3.1334	95266	18
43	30431	31946	3.1303	95257	17
44	30459	31978	3.1271	95248	16
45	30486	32010	3.1240	95240	15
46	30514	32042	3.1209	95231	14
47	30542	32074	3.1178	95222	13
48	30570	32106	3.1146	95213	12
49	30597	32139	3.1115	95204	11
50	30625	32171	3.1084	95195	10
51	30653	32203	3.1053	95186	9
52	30680	32235	3.1022	95177	8
53	30708	32267	3.0991	95168	7
54	30736	32299	3.0961	95159	6
55	30763	32331	3.0930	95150	5
56	30791	32363	3.0899	95142	4
57	30819	32396	3.0868	95133	3
58	30846	32428	3.0838	95124	2
59	30874	32460	3.0807	95115	1
60	30902	32492	3.0777	95106	0
′	Cos	Ctn	Tan	Sin	′

72°

18°

′	Sin	Tan	Ctn	Cos	′
0	.30902	.32492	3.0777	.95106	60
1	.30929	.32524	3.0746	.95097	59
2	.30957	.32556	3.0716	.95088	58
3	.30985	.32588	3.0686	.95079	57
4	.31012	.32621	3.0655	.95070	56
5	.31040	.32653	3.0625	.95061	55
6	.31068	.32685	3.0595	.95052	54
7	.31095	.32717	3.0565	.95043	53
8	.31123	.32749	3.0535	.95033	52
9	.31151	.32782	3.0505	.95024	51
10	.31178	.32814	3.0475	.95015	50
11	.31206	.32846	3.0445	.95006	49
12	.31233	.32878	3.0415	.94997	48
13	.31261	.32911	3.0385	.94988	47
14	.31289	.32943	3.0356	.94979	46
15	.31316	.32975	3.0326	.94970	45
16	.31344	.33007	3.0296	.94961	44
17	.31372	.33040	3.0267	.94952	43
18	.31399	.33072	3.0237	.94943	42
19	.31427	.33104	3.0208	.94933	41
20	.31454	.33136	3.0178	.94924	40
21	.31482	.33169	3.0149	.94915	39
22	.31510	.33201	3.0120	.94906	38
23	.31537	.33233	3.0090	.94897	37
24	.31565	.33266	3.0061	.94888	36
25	.31593	.33298	3.0032	.94878	35
26	.31620	.33330	3.0003	.94869	34
27	.31648	.33363	2.9974	.94860	33
28	.31675	.33395	2.9945	.94851	32
29	.31703	.33427	2.9916	.94842	31
30	.31730	.33460	2.9887	.94832	30
31	.31758	.33492	2.9858	.94823	29
32	.31786	.33524	2.9829	.94814	28
33	.31813	.33557	2.9800	.94805	27
34	.31841	.33589	2.9772	.94795	26
35	.31868	.33621	2.9743	.94786	25
36	.31896	.33654	2.9714	.94777	24
37	.31923	.33686	2.9686	.94768	23
38	.31951	.33718	2.9657	.94758	22
39	.31979	.33751	2.9629	.94749	21
40	.32006	.33783	2.9600	.94740	20
41	.32034	.33816	2.9572	.94730	19
42	.32061	.33848	2.9544	.94721	18
43	.32089	.33881	2.9515	.94712	17
44	.32116	.33913	2.9487	.94702	16
45	.32144	.33945	2.9459	.94693	15
46	.32171	.33978	2.9431	.94684	14
47	.32199	.34010	2.9403	.94674	13
48	.32227	.34043	2.9375	.94665	12
49	.32254	.34075	2.9347	.94656	11
50	.32282	.34108	2.9319	.94646	10
51	.32309	.34140	2.9291	.94637	9
52	.32337	.34173	2.9263	.94627	8
53	.32364	.34205	2.9235	.94618	7
54	.32392	.34238	2.9208	.94609	6
55	.32419	.34270	2.9180	.94599	5
56	.32447	.34303	2.9152	.94590	4
57	.32474	.34335	2.9125	.94580	3
58	.32502	.34368	2.9097	.94571	2
59	.32529	.34400	2.9070	.94561	1
60	.32557	.34433	2.9042	.94552	0
′	Cos	Ctn	Tan	Sin	′

71°

19°

′	Sin	Tan	Ctn	Cos	′
0	.32557	.34433	2.9042	.94552	60
1	.32584	.34465	2.9015	.94542	59
2	.32612	.34498	2.8987	.94533	58
3	.32639	.34530	2.8960	.94523	57
4	.32667	.34563	2.8933	.94514	56
5	.32694	.34596	2.8905	.94504	55
6	.32722	.34628	2.8878	.94495	54
7	.32749	.34661	2.8851	.94485	53
8	.32777	.34693	2.8824	.94476	52
9	.32804	.34726	2.8797	.94466	51
10	.32832	.34758	2.8770	.94457	50
11	.32859	.34791	2.8743	.94447	49
12	.32887	.34824	2.8716	.94438	48
13	.32914	.34856	2.8689	.94428	47
14	.32942	.34889	2.8662	.94418	46
15	.32969	.34922	2.8636	.94409	45
16	.32997	.34954	2.8609	.94399	44
17	.33024	.34987	2.8582	.94390	43
18	.33051	.35020	2.8556	.94380	42
19	.33079	.35052	2.8529	.94370	41
20	.33106	.35085	2.8502	.94361	40
21	.33134	.35118	2.8476	.94351	39
22	.33161	.35150	2.8449	.94342	38
23	.33189	.35183	2.8423	.94332	37
24	.33216	.35216	2.8397	.94322	36
25	.33244	.35248	2.8370	.94313	35
26	.33271	.35281	2.8344	.94303	34
27	.33298	.35314	2.8318	.94293	33
28	.33326	.35346	2.8291	.94284	32
29	.33353	.35379	2.8265	.94274	31
30	.33381	.35412	2.8239	.94264	30
31	.33408	.35445	2.8213	.94254	29
32	.33436	.35477	2.8187	.94245	28
33	.33463	.35510	2.8161	.94235	27
34	.33490	.35543	2.8135	.94225	26
35	.33518	.35576	2.8109	.94215	25
36	.33545	.35608	2.8083	.94206	24
37	.33573	.35641	2.8057	.94196	23
38	.33600	.35674	2.8032	.94186	22
39	.33627	.35707	2.8006	.94176	21
40	.33655	.35740	2.7980	.94167	20
41	.33682	.35772	2.7955	.94157	19
42	.33710	.35805	2.7929	.94147	18
43	.33737	.35838	2.7903	.94137	17
44	.33764	.35871	2.7878	.94127	16
45	.33792	.35904	2.7852	.94118	15
46	.33819	.35937	2.7827	.94108	14
47	.33846	.35969	2.7801	.94098	13
48	.33874	.36002	2.7776	.94088	12
49	.33901	.36035	2.7751	.94078	11
50	.33929	.36068	2.7725	.94068	10
51	.33956	.36101	2.7700	.94058	9
52	.33983	.36134	2.7675	.94049	8
53	.34011	.36167	2.7650	.94039	7
54	.34038	.36199	2.7625	.94029	6
55	.34065	.36232	2.7600	.94019	5
56	.34093	.36265	2.7575	.94009	4
57	.34120	.36298	2.7550	.93999	3
58	.34147	.36331	2.7525	.93989	2
59	.34175	.36364	2.7500	.93979	1
60	.34202	.36397	2.7475	.93969	0
′	Cos	Ctn	Tan	Sin	′

70°

20°

′	Sin	Tan	Ctn	Cos	′
0	.34202	36397	2.7475	93969	60
1	34229	36430	2.7450	93959	59
2	34257	36463	2.7425	93949	58
3	.34284	36496	2.7400	93939	57
4	34311	36529	2.7376	93929	56
5	34339	36562	2.7351	93919	55
6	34366	.36595	2.7326	93909	54
7	34393	36628	2.7302	93899	53
8	34421	36661	2.7277	93889	52
9	34448	36694	2.7253	93879	51
10	34475	36727	2.7228	93869	50
11	34503	36760	2.7204	93859	49
12	34530	36793	2.7179	93849	48
13	34557	36826	2.7155	93839	47
14	34584	36859	2.7130	93829	46
15	34612	36892	2.7106	93819	45
16	34639	36925	2.7082	93809	44
17	34666	36958	2.7058	93799	43
18	34694	36991	2.7034	93789	42
19	34721	37024	2.7009	93779	41
20	34748	37057	2.6985	93769	40
21	34775	.37090	2.6961	93759	39
22	34803	37123	2.6937	93748	38
23	34830	37157	2.6913	93738	37
24	34857	37190	2.6889	93728	36
25	34884	37223	2.6865	93718	35
26	34912	37256	2.6841	93708	34
27	34939	37289	2.6818	93698	33
28	34966	37322	2.6794	93688	32
29	34993	37355	2.6770	93677	31
30	35021	37388	2.6746	93667	30
31	35048	37422	2.6723	93657	29
32	35075	37455	2.6699	93647	28
33	35102	37488	2.6675	93637	27
34	35130	37521	2.6652	93626	26
35	35157	37554	2.6628	93616	25
36	35184	37588	2.6605	93606	24
37	35211	37621	2.6581	93596	23
38	35239	37654	2.6558	93585	22
39	35266	37687	2.6534	93575	21
40	35293	37720	2.6511	93565	20
41	35320	37754	2.6488	93555	19
42	35347	37787	2.6464	93544	18
43	35375	37820	2.6441	93534	17
44	35402	37853	2.6418	93524	16
45	35429	37887	2.6395	93514	15
46	.35456	37920	2.6371	93503	14
47	35484	37953	2.6348	93493	13
48	35511	37986	2.6325	93483	12
49	35538	38020	2.6302	93472	11
50	35565	38053	2.6279	93462	10
51	35592	38086	2.6256	93452	9
52	35619	38120	2.6233	93441	8
53	35647	38153	2.6210	93431	7
54	35674	38186	2.6187	93420	6
55	35701	38220	2.6165	93410	5
56	35728	38253	2.6142	93400	4
57	35755	38286	2.6119	93389	3
58	35782	38320	2.6096	93379	2
59	35810	38353	2.6074	93368	1
60	35837	38386	2.6051	93358	0
′	Cos	Ctn	Tan	Sin	′

69°

21°

′	Sin	Tan	Ctn	Cos	′
0	35837	38386	2.6051	93358	60
1	35864	38420	2.6028	93348	59
2	35891	38453	2.6006	93337	58
3	35918	38487	2.5983	93327	57
4	35945	38520	2.5961	93316	56
5	35973	38553	2.5938	93306	55
6	36000	38587	2.5916	93295	54
7	36027	38620	2.5893	93285	53
8	36054	38654	2.5871	93274	52
9	36081	38687	2.5848	93264	51
10	36108	38721	2.5826	93253	50
11	36135	38754	2.5804	93243	49
12	36162	38787	2.5782	93232	48
13	36190	38821	2.5759	93222	47
14	36217	38854	2.5737	93211	46
15	36244	38888	2.5715	93201	45
16	36271	38921	2.5693	93190	44
17	36298	38955	2.5671	93180	43
18	.36325	38988	2.5649	93169	42
19	36352	39022	2.5627	93159	41
20	36379	39055	2.5605	93148	40
21	36406	39089	2.5583	93137	39
22	.36434	39122	2.5561	93127	38
23	36461	39156	2.5539	93116	37
24	36488	39190	2.5517	93106	36
25	36515	39223	2.5495	93095	35
26	36542	39257	2.5473	93084	34
27	36569	39290	2.5452	93074	33
28	36596	39324	2.5430	93063	32
29	36623	39357	2.5408	93052	31
30	36650	39391	2.5386	93042	30
31	36677	39425	2.5365	93031	29
32	36704	39458	2.5343	93020	28
33	36731	39492	2.5322	93010	27
34	36758	39526	2.5300	92999	26
35	36785	39559	2.5279	92988	25
36	36812	39593	2.5257	92978	24
37	36839	39626	2.5236	92967	23
38	36867	39660	2.5214	92956	22
39	36894	39694	2.5193	92945	21
40	36921	39727	2.5172	92935	20
41	.36948	39761	2.5150	92924	19
42	36975	39795	2.5129	92913	18
43	37002	39829	2.5108	92902	17
44	37029	39862	2.5086	92892	16
45	37056	39896	2.5065	92881	15
46	37083	39930	2.5044	92870	14
47	.37110	.39963	2.5023	92859	13
48	37137	39997	2.5002	92849	12
49	37164	40031	2.4981	92838	11
50	37191	40065	2.4960	92827	10
51	37218	40098	2.4939	92816	9
52	37245	40132	2.4918	92805	8
53	37272	40166	2.4897	92794	7
54	37299	40200	2.4876	92784	6
55	37326	40234	2.4855	92773	5
56	37353	40267	2.4834	92762	4
57	37380	40301	2.4813	92751	3
58	37407	40335	2.4792	92740	2
59	37434	40369	2.4772	92729	1
60	37461	40403	2.4751	92718	0
′	Cos	Ctn	Tan	Sin	′

68°

22°

′	Sin	Tan	Ctn	Cos	′
0	.37461	.40403	2 4751	.92718	60
1	.37488	.40436	2.4730	.92707	59
2	.37515	.40470	2 4709	.92697	58
3	.37542	.40504	2 4689	.92686	57
4	.37569	.40538	2.4668	.92675	56
5	.37595	.40572	2 4648	.92664	55
6	.37622	.40606	2 4627	.92653	54
7	37649	.40640	2 4606	92642	53
8	.37676	.40674	2 4586	.92631	52
9	37703	.40707	2 4566	.92620	51
10	.37730	.40741	2 4545	92609	50
11	.37757	.40775	2 4525	92598	49
12	.37784	.40809	2 4504	.92587	48
13	.37811	.40843	2 4484	.92576	47
14	.37838	.40877	2 4464	.92565	46
15	37865	.40911	2 4443	.92554	45
16	.37892	.40945	2.4423	.92543	44
17	.37919	.40979	2.4403	.92532	43
18	.37946	.41013	2 4383	92521	42
19	.37973	.41047	2.4362	.92510	41
20	.37999	.41081	2 4342	.92499	40
21	.38026	.41115	2 4322	92488	39
22	38053	41149	2 4302	92477	38
23	38080	41183	2.4282	92466	37
24	.38107	.41217	2 4262	.92455	36
25	.38134	.41251	2 4242	92444	35
26	.38161	.41285	2 4222	.92432	34
27	.38188	.41319	2.4202	.92421	33
28	.38215	.41353	2 4182	.92410	32
29	.38241	.41387	2.4162	.92399	31
30	.38268	.41421	2 4142	.92388	30
31	.38295	.41455	2 4122	92377	29
32	.38322	.41490	2 4102	.92366	28
33	.38349	41524	2.4083	.92355	27
34	.38376	.41558	2.4063	.92343	26
35	.38403	.41592	2 4043	.92332	25
36	.38430	.41626	2 4023	.92321	24
37	.38456	.41660	2 4004	.92310	23
38	.38483	.41694	2 3984	.92299	22
39	.38510	.41728	2.3964	.92287	21
40	.38537	.41763	2 3945	92276	20
41	.38564	.41797	2 3925	.92265	19
42	.38591	.41831	2.3906	.92254	18
43	.38617	.41865	2 3886	.92243	17
44	.38644	.41899	2 3867	.92231	16
45	38671	.41933	2 3847	92220	15
46	38698	.41968	2 3828	92209	14
47	.38725	.42002	2 3808	92198	13
48	.38752	.42036	2 3789	92186	12
49	.38778	.42070	2 3770	.92175	11
50	.38805	.42105	2 3750	92164	10
51	.38832	.42139	2 3731	.92152	9
52	.38859	.42173	2 3712	.92141	8
53	.38886	.42207	2 3693	.92130	7
54	.38912	.42242	2 3673	.92119	6
55	.38939	.42276	2 3654	.92107	5
56	.38966	.42310	2 3635	.92096	4
57	38993	.42345	2 3616	.92085	3
58	.39020	.42379	2 3597	92073	2
59	.39046	.42413	2 3578	.92062	1
60	.39073	.42447	2 3559	.92050	0
′	Cos	Ctn	Tan	Sin	′

67°

23°

′	Sin	Tan	Ctn	Cos	′
0	39073	42447	2 3559	92050	60
1	.39100	42482	2 3539	92039	59
2	39127	42516	2 3520	92028	58
3	39153	42551	2 3501	.92016	57
4	.39180	42585	2 3483	.92005	56
5	39207	42619	2 3464	91994	55
6	.39234	42654	2 3445	91982	54
7	.39260	42688	2 3426	91971	53
8	39287	42722	2 3407	91959	52
9	39314	42757	2 3388	91948	51
10	39341	42791	2 3369	91936	50
11	39367	42826	2 3351	91925	49
12	39394	42860	2 3332	91914	48
13	39421	42894	2 3313	91902	47
14	39448	.42929	2 3294	91891	46
15	39474	42963	2 3276	91879	45
16	39501	42998	2 3257	91868	44
17	39528	43032	2 3238	91856	43
18	39555	43067	2 3220	91845	42
19	39581	43101	2 3201	91833	41
20	39608	43136	2 3183	91822	40
21	39635	43170	2 3164	91810	39
22	39661	43205	2 3146	91799	38
23	39688	43239	2 3127	91787	37
24	39715	43274	2 3109	91775	36
25	39741	43308	2 3090	91764	35
26	39768	43343	2 3072	91752	34
27	39795	43378	2 3053	91741	33
28	39822	43412	2 3035	91729	32
29	39848	43447	2 3017	91718	31
30	39875	43481	2 2998	91706	30
31	39902	43516	2 2980	91694	29
32	39928	43550	2 2962	91683	28
33	39955	.43585	2 2944	91671	27
34	39982	.43620	2 2925	91660	26
35	40008	43654	2 2907	91648	25
36	40035	.43689	2 2889	.91636	24
37	40062	43724	2 2871	.91625	23
38	.40088	43758	2 2853	91613	22
39	.40115	.43793	2 2835	91601	21
40	40141	43828	2 2817	91590	20
41	.40168	43862	2 2799	91578	19
42	.40195	43897	2 2781	91566	18
43	.40221	43932	2 2763	.91555	17
44	40248	.43966	2.2745	.91543	16
45	.40275	44001	2 2727	.91531	15
46	40301	44036	2 2709	.91519	14
47	40328	44071	2 2691	.91508	13
48	40355	44105	2 2673	.91496	12
49	40381	44140	2.2655	.91484	11
50	40408	44175	2 2637	91472	10
51	40434	44210	2 2620	91461	9
52	40461	44244	2 2602	91449	8
53	40488	44279	2 2584	91437	7
54	.40514	44314	2 2566	.91425	6
55	40541	44349	2 2549	.91414	5
56	40567	.44384	2 2531	91402	4
57	40594	.44418	2 2513	91390	3
58	.40621	44453	2 2496	91378	2
59	40647	44488	2 2478	.91366	1
60	40674	.44523	2 2460	.91355	0
′	Cos	Ctn	Tan	Sin	′

66°

24°

′	Sin	Tan	Ctn	Cos	′
0	40674	44523	2 2460	91355	60
1	40700	44558	2 2443	91343	59
2	40727	44593	2 2425	91331	58
3	40753	44627	2 2408	91319	57
4	40780	44662	2 2390	91307	56
5	40806	44697	2 2373	91295	55
6	40833	44732	2 2355	91283	54
7	40860	44767	2 2338	91272	53
8	40886	44802	2 2320	91260	52
9	40913	44837	2 2303	91248	51
10	40939	44872	2 2286	91236	50
11	40966	44907	2 2268	91224	49
12	40992	44942	2 2251	91212	48
13	41019	44977	2 2234	91200	47
14	41045	45012	2 2216	91188	46
15	41072	45047	2 2199	91176	45
16	41098	45082	2 2182	91164	44
17	41125	45117	2 2165	91152	43
18	41151	45152	2 2148	91140	42
19	41178	45187	2 2130	91128	41
20	41204	45222	2 2113	91116	40
21	41231	45257	2 2096	91104	39
22	41257	45292	2 2079	91092	38
23	41284	45327	2 2062	91080	37
24	41310	45362	2 2045	91068	36
25	41337	45397	2 2028	91056	35
26	41363	45432	2 2011	91044	34
27	41390	45467	2 1994	91032	33
28	41416	45502	2 1977	91020	32
29	41443	45538	2 1960	91008	31
30	41469	45573	2 1943	90996	30
31	41496	45608	2 1926	90984	29
32	41522	45643	2 1909	90972	28
33	41549	45678	2 1892	90960	27
34	41575	45713	2 1876	90948	26
35	41602	45748	2 1859	90936	25
36	41628	45784	2 1842	90924	24
37	41655	45819	2 1825	90911	23
38	41681	45854	2 1808	90899	22
39	41707	45889	2 1792	90887	21
40	41734	45924	2 1775	90875	20
41	41760	45960	2 1758	90863	19
42	41787	45995	2 1742	90851	18
43	41813	46030	2 1725	90839	17
44	41840	46065	2 1708	90826	16
45	41866	46101	2 1692	90814	15
46	41892	46136	2 1675	90802	14
47	41919	46171	2 1659	90790	13
48	41945	46206	2 1642	90778	12
49	41972	46242	2 1625	90766	11
50	41998	46277	2 1609	90753	10
51	42024	46312	2 1592	90741	9
52	42051	46348	2 1576	90729	8
53	42077	46383	2 1560	90717	7
54	42104	46418	2 1543	90704	6
55	42130	46454	2 1527	90692	5
56	42156	46489	2 1510	90680	4
57	42183	46525	2 1494	90668	3
58	42209	46560	2 1478	90655	2
59	42235	46595	2 1461	90643	1
60	42262	46631	2 1445	90631	
′	Cos	Ctn	Tan	Sin	′

65°

25°

′	Sin	Tan	Ctn	Cos	′
0	42262	46631	2 1445	90631	60
1	42288	46666	2 1429	90618	59
2	42315	46702	2 1413	90606	58
3	42341	46737	2 1396	90594	57
4	42367	46772	2 1380	90582	56
5	42394	46808	2 1364	90569	55
6	42420	46843	2 1348	90557	54
7	42446	46879	2 1332	90545	53
8	42473	46914	2 1315	90532	52
9	42499	46950	2 1299	90520	51
10	42525	46985	2 1283	90507	50
11	42552	47021	2 1267	90495	49
12	42578	47056	2 1251	90483	48
13	42604	47092	2 1235	90470	47
14	42631	47128	2 1219	90458	46
15	42657	47163	2 1203	90446	45
16	42683	47199	2 1187	90433	44
17	42709	47234	2 1171	90421	43
18	42736	47270	2 1155	90408	42
19	42762	47305	2 1139	90396	41
20	42788	47341	2 1123	90383	40
21	42815	47377	2 1107	90371	39
22	42841	47412	2 1092	90358	38
23	42867	47448	2 1076	90346	37
24	42894	47483	2 1060	90334	36
25	42920	47519	2 1044	90321	35
26	42946	47555	2 1028	90309	34
27	42972	47590	2 1013	90296	33
28	42999	47626	2 0997	90284	32
29	43025	47662	2 0981	90271	31
30	43051	47698	2 0965	90259	30
31	43077	47733	2 0950	90246	29
32	43104	47769	2 0934	90233	28
33	43130	47805	2 0918	90221	27
34	43156	47840	2 0903	90208	26
35	43182	47876	2 0887	90196	25
36	43209	47912	2 0872	90183	24
37	43235	47948	2 0856	90171	23
38	43261	47984	2 0840	90158	22
39	43287	48019	2 0825	90146	21
40	43313	48055	2 0809	90133	20
41	43340	48091	2 0794	90120	19
42	43366	48127	2 0778	90108	18
43	43392	48163	2 0763	90095	17
44	43418	48198	2 0748	90082	16
45	43445	48234	2 0732	90070	15
46	43471	48270	2 0717	90057	14
47	43497	48306	2 0701	90045	13
48	43523	48342	2 0686	90032	12
49	43549	48378	2 0671	90019	11
50	43575	48414	2 0655	90007	10
51	43602	48450	2 0640	89994	9
52	43628	48486	2 0625	89981	8
53	43654	48521	2 0609	89968	7
54	43680	48557	2 0594	89956	6
55	43706	48593	2 0579	89943	5
56	43733	48629	2 0564	89930	4
57	43759	48665	2 0549	89918	3
58	43785	48701	2 0533	89905	2
59	43811	48737	2 0518	89892	1
60	43837	48773	2 0503	89879	0
′	Cos	Ctn	Tan	Sin	′

64°

26°

′	Sin	Tan	Ctn	Cos	′
0	.43837	.48773	2 0503	89879	60
1	.43863	.48809	2 0488	89867	59
2	.43889	48845	2 0473	89854	58
3	.43916	48881	2 0458	89841	57
4	.43942	.48917	2 0443	.89828	56
5	.43968	48953	2 0428	89816	55
6	.43994	.48989	2 0413	89803	54
7	44020	.49026	2 0398	89790	53
8	44046	49062	2 0383	89777	52
9	.44072	.49098	2 0368	.89764	51
10	.44098	.49134	2 0353	89752	50
11	.44124	.49170	2 0338	.89739	49
12	.44151	.49206	2 0323	89726	48
13	.44177	.49242	2 0308	89713	47
14	.44203	.49278	2 0293	89700	46
15	.44229	.49315	2 0278	89687	45
16	.44255	.49351	2 0263	.89674	44
17	.44281	.49387	2 0248	89662	43
18	.44307	.49423	2 0233	.89649	42
19	.44333	.49459	2 0219	89636	41
20	44359	.49495	2 0204	.89623	40
21	.44385	.49532	2 0189	.89610	39
22	.44411	.49568	2 0174	.89597	38
23	.44437	.49604	2 0160	.89584	37
24	.44464	.49640	2 0145	.89571	36
25	44490	.49677	2 0130	89558	35
26	.44516	.49713	2 0115	.89545	34
27	.44542	49749	2.0101	89532	33
28	.44568	.49786	2 0086	89519	32
29	.44594	.49822	2 0072	89506	31
30	.44620	.49858	2 0057	.89493	30
31	.44646	.49894	2 0042	.89480	29
32	.44672	49931	2 0028	.89467	28
33	.44698	49967	2 0013	.89454	27
34	.44724	.50004	1 9999	89441	26
35	.44750	.50040	1 9984	.89428	25
36	.44776	.50076	1 9970	.89415	24
37	.44802	.50113	1 9955	.89402	23
38	.44828	.50149	1 9941	89389	22
39	.44854	.50185	1.9926	89376	21
40	44880	50222	1 9912	.89363	20
41	.44906	.50258	1.9897	.89350	19
42	.44932	50295	1 9883	89337	18
43	.44958	.50331	1.9868	89324	17
44	.44984	.50368	1 9854	.89311	16
45	45010	.50404	1 9840	89298	15
46	45036	.50441	1 9825	89285	14
47	45062	.50477	1.9811	.89272	13
48	.45088	50514	1 9797	.89259	12
49	.45114	.50550	1.9782	.89245	11
50	.45140	.50587	1 9768	89232	10
51	45166	.50623	1 9754	.89219	9
52	45192	.50660	1.9740	.89206	8
53	45218	.50696	1 9725	89193	7
54	.45243	.50733	1 9711	.89180	6
55	.45269	.50769	1.9697	.89167	5
56	45295	50806	1 9683	.89153	4
57	.45321	.50843	1 9669	.89140	3
58	.45347	.50879	1 9654	.89127	2
59	.45373	.50916	1 9640	.89114	1
60	.45399	.50953	1 9626	.89101	0
′	Cos	Ctn	Tan	Sin	′

63°

27°

′	Sin	Tan	Ctn	Cos	′
0	45399	.50953	1 9626	89101	60
1	45425	.50989	1 9612	89087	59
2	45451	51026	1 9598	89074	58
3	45477	.51063	1 9584	89061	57
4	45503	51099	1 9570	89048	56
5	45529	51136	1 9556	89035	55
6	45554	51173	1 9542	89021	54
7	45580	.51209	1 9528	89008	53
8	45606	51246	1 9514	88995	52
9	45632	51283	1 9500	88981	51
10	45658	51319	1 9486	88968	50
11	.45684	51356	1 9472	88955	49
12	45710	51393	1 9458	88942	48
13	45736	51430	1 9444	88928	47
14	45762	.51467	1 9430	88915	46
15	45787	51503	1 9416	.88902	45
16	45813	51540	1 9402	88888	44
17	45839	51577	1 9388	88875	43
18	45865	51614	1 9375	88862	42
19	45891	51651	1 9361	88848	41
20	45917	51688	1 9347	88835	40
21	45942	51724	1 9333	88822	39
22	45968	51761	1 9319	88808	38
23	45994	51798	1 9306	88795	37
24	.46020	.51835	1 9292	88782	36
25	46046	.51872	1 9278	88768	35
26	46072	.51909	1.9265	88755	34
27	46097	51946	1 9251	88741	33
28	46123	.51983	1.9237	88728	32
29	46149	52020	1 9223	88715	31
30	46175	.52057	1.9210	.88701	30
31	46201	.52094	1 9196	88688	29
32	.46226	52131	1 9183	88674	28
33	.46252	.52168	1 9169	88661	27
34	46278	.52205	1.9155	.88647	26
35	46304	.52242	1 9142	.88634	25
36	46330	.52279	1 9128	.88620	24
37	46355	.52316	1 9115	88607	23
38	46381	.52353	1 9101	88593	22
39	46407	.52390	1 9088	.88580	21
40	.46433	.52427	1.9074	88566	20
41	.46458	.52464	1 9061	88553	19
42	.46484	.52501	1 9047	88539	18
43	46510	.52538	1 9034	88526	17
44	46536	52575	1 9020	88512	16
45	.46561	.52613	1 9007	88499	15
46	.46587	.52650	1 8993	88485	14
47	.46613	52687	1 8980	88472	13
48	46639	52724	1 8967	88458	12
49	46664	.52761	1 8953	88445	11
50	46690	.52798	1 8940	88431	10
51	46716	.52836	1 8927	.88417	9
52	.46742	.52873	1 8913	.88404	8
53	.46767	.52910	1 8900	.88390	7
54	.46793	.52947	1 8887	.88377	6
55	.46819	52985	1.8873	.88363	5
56	46844	.53022	1 8860	.88349	4
57	.46870	.53059	1.8847	.88336	3
58	.46896	.53096	1 8834	.88322	2
59	.46921	.53134	1 8820	.88308	1
60	.46947	.53171	1 8807	.88295	0
′	Cos	Ctn	Tan	Sin	′

62°

28°

′	S n	Tan	Ctn	Cos	′
0	46947	53171	1 8807	88295	60
1	46973	53208	1 8794	88281	59
2	46999	53246	1 8781	88267	58
3	47024	53283	1 8768	88254	57
4	47050	53320	1 8755	88240	56
5	47076	53358	1 8741	88226	55
6	47101	53395	1 8728	88213	54
7	47127	53432	1 8715	88199	53
8	47153	53470	1 8702	88185	52
9	47178	53507	1 8689	88172	51
10	47204	53545	1 8676	88158	50
11	47229	53582	1 8663	88144	49
12	47255	53620	1 8650	88130	48
13	47281	53657	1 8637	88117	47
14	47306	53694	1 8624	88103	46
15	47332	53732	1 8611	88089	45
16	47358	53769	1 8598	88075	44
17	47383	53807	1 8585	88062	43
18	47409	53844	1 8572	88048	42
19	47434	53882	1 8559	88034	41
20	47460	53920	1 8546	88020	40
21	47486	53957	1 8533	88006	39
22	47511	53995	1 8520	87993	38
23	47537	54032	1 8507	87979	37
24	47562	54070	1 8495	87965	36
25	47588	54107	1 8482	87951	35
26	47614	54145	1 8469	87937	34
27	47639	54183	1 8456	87923	33
28	47665	54220	1 8443	87909	32
29	47690	54258	1 8430	87896	31
30	47716	54296	1 8418	87882	30
31	47741	54333	1 8405	87868	29
32	47767	54371	1 8392	87854	28
33	47793	54409	1 8379	87840	27
34	47818	54446	1 8367	87826	26
35	47844	54484	1 8354	87812	25
36	47869	54522	1 8341	87798	24
37	47895	54560	1 8329	87784	23
38	47920	54597	1 8316	87770	22
39	47946	54635	1 8303	87756	21
40	47971	54673	1 8291	87743	20
41	47997	54711	1 8278	87729	19
42	48022	54748	1 8265	87715	18
43	48048	54786	1 8253	87701	17
44	48073	54824	1 8240	87687	16
45	48099	54862	1 8228	87673	15
46	48124	54900	1 8215	87659	14
47	48150	54938	1 8202	87645	13
48	48175	54975	1 8190	87631	12
49	48201	55013	1 8177	87617	11
50	48226	55051	1 8165	87603	10
51	48252	55089	1 8152	87589	9
52	48277	55127	1 8140	87575	8
53	48303	55165	1 8127	87561	7
54	48328	55203	1 8115	87546	6
55	48354	55241	1 8103	87532	5
56	48379	55279	1 8090	87518	4
57	48405	55317	1 8078	87504	3
58	48430	55355	1 8065	87490	2
59	48456	55393	1 8053	87476	1
60	48481	55431	1 8040	87462	0
′	Cos	Ctn	Tan	S n	′

61°

29°

′	S n	Tan	Ctn	Cos	′
0	48481	55431	1 8040	87462	60
1	48506	55469	1 8028	87448	59
2	48532	55507	1 8016	87434	58
3	48557	55545	1 8003	87420	57
4	48583	55583	1 7991	87406	56
5	48608	55621	1 7979	87391	55
6	48634	55659	1 7966	87377	54
7	48659	55697	1 7954	87363	53
8	48684	55736	1 7942	87349	52
9	48710	55774	1 7930	87335	51
10	48735	55812	1 7917	87321	50
11	48761	55850	1 7905	87306	49
12	48786	55888	1 7893	87292	48
13	48811	55926	1 7881	87278	47
14	48837	55964	1 7868	87264	46
15	48862	56003	1 7856	87250	45
16	48888	56041	1 7844	87235	44
17	48913	56079	1 7832	87221	43
18	48938	56117	1 7820	87207	42
19	48964	56156	1 7808	87193	41
20	48989	56194	1 7796	87178	40
21	49014	56232	1 7783	87164	39
22	49040	56270	1 7771	87150	38
23	49065	56309	1 7759	87136	37
24	49090	56347	1 7747	87121	36
25	49116	56385	1 7735	87107	35
26	49141	56424	1 7723	87093	34
27	49166	56462	1 7711	87079	33
28	49192	56501	1 7699	87064	32
29	49217	56539	1 7687	87050	31
30	49242	56577	1 7675	87036	30
31	49268	56616	1 7663	87021	29
32	49293	56654	1 7651	87007	28
33	49318	56693	1 7639	86993	27
34	49344	56731	1 7627	86978	26
35	49369	56769	1 7615	86964	25
36	49394	56808	1 7603	86949	24
37	49419	56846	1 7591	86935	23
38	49445	56885	1 7579	86921	22
39	49470	56923	1 7567	86906	21
40	49495	56962	1 7556	86892	20
41	49521	57000	1 7544	86878	19
42	49546	57039	1 7532	86863	18
43	49571	57078	1 7520	86849	17
44	49596	57116	1 7508	86834	16
45	49622	57155	1 7496	86820	15
46	49647	57193	1 7485	86805	14
47	49672	57232	1 7473	86791	13
48	49697	57271	1 7461	86777	12
49	49723	57309	1 7449	86762	11
50	49748	57348	1 7437	86748	10
51	49773	57386	1 7426	86733	9
52	49798	57425	1 7414	86719	8
53	49824	57464	1 7402	86704	7
54	49849	57503	1 7391	86690	6
55	49874	57541	1 7379	86675	5
56	49899	57580	1 7367	86661	4
57	49924	57619	1 7355	86646	3
58	49950	57657	1 7344	86632	2
59	49975	57696	1 7332	86617	1
60	50000	57735	1 7321	86603	0
′	Cos	Ctn	Tan	S n	′

60°

30°

′	Sin	Tan	Ctn	Cos	′
0	50000	57735	1 7321	86603	60
1	50025	57774	1 7309	86588	59
2	50050	57813	1 7297	86573	58
3	50076	57851	1 7286	86559	57
4	50101	57890	1 7274	86544	56
5	50126	57929	1 7262	86530	55
6	50151	57968	1 7251	86515	54
7	50176	58007	1 7239	86501	53
8	50201	58046	1 7228	86486	52
9	50227	58085	1 7216	86471	51
10	50252	58124	1 7205	86457	50
11	50277	58162	1 7193	86442	49
12	50302	48201	1 7182	86427	48
13	50327	58240	1 7170	86413	47
14	50352	58279	1 7159	86398	46
15	50377	58318	1 7147	86384	45
16	50403	58357	1 7136	86369	44
17	50428	58396	1 7124	86354	43
18	50453	58435	1 7113	86340	42
19	50478	58474	1 7102	86325	41
20	50503	58513	1 7090	86310	40
21	50528	58552	1 7079	86295	39
22	50553	58591	1 7067	86281	38
23	50578	58631	1 7056	86266	37
24	50603	58670	1 7045	86251	36
25	50628	58709	1 7033	86237	35
26	50654	58748	1 7022	86222	34
27	50679	58787	1 7011	86207	33
28	50704	58826	1 6999	86192	32
29	50729	58865	1 6988	86178	31
30	50754	58905	1 6977	86163	30
31	50779	58944	1 6965	86148	29
32	50804	58983	1 6954	86133	28
33	50829	59022	1 6943	86119	27
34	50854	59061	1 6932	86104	26
35	50879	59101	1 6920	86089	25
36	50904	59140	1 6909	86074	24
37	50929	59179	1 6898	86059	23
38	50954	59218	1 6887	86045	22
39	50979	59258	1 6875	86030	21
40	51004	59297	1 6864	86015	20
41	51029	59336	1 6853	86000	19
42	51054	59376	1 6842	85985	18
43	51079	59415	1 6831	85970	17
44	51104	59454	1 6820	85956	16
45	51129	59494	1 6808	85941	15
46	51154	59533	1 6797	85926	14
47	51179	59573	1 6786	85911	13
48	51204	59612	1 6775	85896	12
49	51229	59651	1 6764	85881	11
50	51254	59691	1 6753	85866	10
51	51279	59730	1 6742	85851	9
52	51304	59770	1 6731	85836	8
53	51329	59809	1 6720	85821	7
54	51354	59849	1 6709	85806	6
55	51379	59888	1 6698	85792	5
56	51404	59928	1 6687	85777	4
57	51429	59967	1 6676	85762	3
58	51454	60007	1 6665	85747	2
59	51479	60046	1 6654	85732	1
60	51504	60086	1 6643	85717	0
′	Cos	Ctn	Tan	Sin	′

59°

31°

′	Sin	Tan	Ctn	Cos	′
0	51504	60086	1 6643	85717	60
1	51529	60126	1 6632	85702	59
2	51554	60165	1 6621	85687	58
3	51579	60205	1 6610	85672	57
4	51604	60245	1 6599	85657	56
5	51628	60284	1 6588	85642	55
6	51653	60324	1 6577	85627	54
7	51678	60364	1 6566	85612	53
8	51703	60403	1 6555	85597	52
9	51728	60443	1 6545	85582	51
10	51753	60483	1 6534	85567	50
11	51778	60522	1 6523	85551	49
12	51803	60562	1 6512	85536	48
13	51828	60602	1 6501	85521	47
14	51852	60642	1 6490	85506	46
15	51877	60681	1 6479	85491	45
16	51902	60721	1 6469	85476	44
17	51927	60761	1 6458	85461	43
18	51952	60801	1 6447	85446	42
19	51977	60841	1 6436	85431	41
20	52002	60881	1 6426	85416	40
21	52026	60921	1 6415	85401	39
22	52051	60960	1 6404	85385	38
23	52076	61000	1 6393	85370	37
24	52101	61040	1 6383	85355	36
25	52126	61080	1 6372	85340	35
26	52151	61120	1 6361	85325	34
27	52175	61160	1 6351	85310	33
28	52200	61200	1 6340	85294	32
29	52225	61240	1 6329	85279	31
30	52250	61280	1 6319	85264	30
31	52275	61320	1 6308	85249	29
32	52299	61360	1 6297	85234	28
33	52324	61400	1 6287	85218	27
34	52349	61440	1 6276	85203	26
35	52374	61480	1 6265	85188	25
36	52399	61520	1 6255	85173	24
37	52423	61561	1 6244	85157	23
38	52448	61601	1 6234	85142	22
39	52473	61641	1 6223	85127	21
40	52498	61681	1 6212	85112	20
41	52522	61721	1 6202	85096	19
42	52547	61761	1 6191	85081	18
43	52572	61801	1 6181	85066	17
44	52597	61842	1 6170	85051	16
45	52621	61882	1 6160	85035	15
46	52646	61922	1 6149	85020	14
47	52671	61962	1 6139	85005	13
48	52696	62003	1 6128	84989	12
49	52720	62043	1 6118	84974	11
50	52745	62083	1 6107	84959	10
51	52770	62124	1 6097	84943	9
52	52794	62164	1 6087	84928	8
53	52819	62204	1 6076	84913	7
54	52844	62245	1 6066	84897	6
55	52869	62285	1 6055	84882	5
56	52893	62325	1 6045	84866	4
57	52918	62366	1 6034	84851	3
58	52943	62406	1 6024	84836	2
59	52967	62446	1 6014	84820	1
60	52992	62487	1 6003	84805	0
′	Cos	Ctn	Tan	Sin	′

58°

32°

′	Sin	Tan	Ctn	Cos	′
0	52992	62487	1.6003	84805	60
1	53017	62527	1.5993	84789	59
2	53041	62568	1.5983	84774	58
3	53066	62608	1.5972	84759	57
4	53091	62649	1.5962	84743	56
5	53115	62689	1.5952	84728	55
6	53140	62730	1.5941	84712	54
7	53164	62770	1.5931	84697	53
8	53189	62811	1.5921	84681	52
9	51214	62852	1.5911	84666	51
10	53238	62892	1.5900	84650	50
11	53263	62933	1.5890	84635	49
12	53288	62973	1.5880	84619	48
13	53312	63014	1.5869	84604	47
14	53337	63055	1.5859	84588	46
15	53361	63095	1.5849	84573	45
16	53386	63136	1.5839	84557	44
17	53411	63177	1.5829	84542	43
18	53435	63217	1.5818	84526	42
19	53460	63258	1.5808	84511	41
20	53484	63299	1.5798	84495	40
21	53509	63340	1.5788	84480	39
22	53534	63380	1.5778	84464	38
23	53558	63421	1.5768	84448	37
24	53583	63462	1.5757	84433	36
25	53607	63503	1.5747	84417	35
26	53632	63544	1.5737	84402	34
27	53656	63584	1.5727	84386	33
28	53681	63625	1.5717	84370	32
29	53705	63666	1.5707	84355	31
30	53730	63707	1.5697	84339	30
31	53754	63748	1.5687	84324	29
32	53779	63789	1.5677	84308	28
33	53804	63830	1.5667	84292	27
34	53828	63871	1.5657	84277	26
35	53853	63912	1.5647	84261	25
36	53877	63953	1.5637	84245	24
37	53902	63994	1.5627	84230	23
38	53926	64035	1.5617	84214	22
39	53951	64076	1.5607	84198	21
40	53975	64117	1.5597	84182	20
41	54000	64158	1.5587	84167	19
42	54024	64199	1.5577	84151	18
43	54049	64240	1.5567	84135	17
44	54073	64281	1.5557	84120	16
45	54097	64322	1.5547	84104	15
46	54122	64363	1.5537	84088	14
47	54146	64404	1.5527	84072	13
48	54171	64446	1.5517	84057	12
49	54195	64487	1.5507	84041	11
50	54220	64528	1.5497	84025	10
51	54244	64569	1.5487	84009	9
52	54269	64610	1.5477	83994	8
53	54293	64652	1.5468	83978	7
54	54317	64693	1.5458	83962	6
55	54342	64734	1.5448	83946	5
56	54366	64775	1.5438	83930	4
57	54391	64817	1.5428	83915	3
58	54415	64858	1.5418	83899	2
59	54440	64899	1.5408	83883	1
60	54464	64941	1.5399	83867	0
′	Cos	Ctn	Tan	Sin	′

57°

33°

′	Sin	Tan	Ctn	Cos	′
0	54464	64941	1.5399	83867	60
1	54488	64982	1.5389	83851	59
2	54513	65024	1.5379	83835	58
3	54537	65065	1.5369	83819	57
4	54561	65106	1.5359	83804	56
5	54586	65148	1.5350	83788	55
6	54610	65189	1.5340	83772	54
7	54635	65231	1.5330	83756	53
8	54659	65272	1.5320	83740	52
9	54683	65314	1.5311	83724	51
10	54708	65355	1.5301	83708	50
11	54732	65397	1.5291	83692	49
12	54756	65438	1.5282	83676	48
13	54781	65480	1.5272	83660	47
14	54805	65521	1.5262	83645	46
15	54829	65563	1.5253	83629	45
16	54854	65604	1.5243	83613	44
17	54878	65646	1.5233	83597	43
18	54902	65688	1.5224	83581	42
19	54927	65729	1.5214	83565	41
20	54951	65771	1.5204	83549	40
21	54975	65813	1.5195	83533	39
22	54999	65854	1.5185	83517	38
23	55024	65896	1.5175	83501	37
24	55048	65938	1.5166	83485	36
25	55072	65980	1.5156	83469	35
26	55097	66021	1.5147	83453	34
27	55121	66063	1.5137	83437	33
28	55145	66105	1.5127	83421	32
29	55169	66147	1.5118	83405	31
30	55194	66189	1.5108	83389	30
31	55218	66230	1.5099	83373	29
32	55242	66272	1.5089	83356	28
33	55266	66314	1.5080	83340	27
34	55291	66356	1.5070	83324	26
35	55315	66398	1.5061	83308	25
36	55339	66440	1.5051	83292	24
37	55363	66482	1.5042	83276	23
38	55388	66524	1.5032	83260	22
39	55412	66566	1.5023	83244	21
40	55436	66608	1.5013	83228	20
41	55460	66650	1.5004	83212	19
42	55484	66692	1.4994	83195	18
43	55509	66734	1.4985	83179	17
44	55533	66776	1.4975	83163	16
45	55557	65818	1.4966	83147	15
46	55581	65960	1.4957	83131	14
47	55605	66902	1.4947	83115	13
48	55630	66944	1.4938	83098	12
49	55654	66986	1.4928	83082	11
50	55678	67028	1.4919	83066	10
51	55702	67071	1.4910	83050	9
52	55726	67113	1.4900	83034	8
53	55750	67155	1.4891	83017	7
54	55775	67197	1.4882	83001	6
55	55799	67239	1.4872	82985	5
56	55823	67282	1.4863	82969	4
57	55847	67324	1.4854	82953	3
58	55871	67366	1.4844	82936	2
59	55895	67409	1.4835	82920	1
60	55919	67451	1.4826	82904	0
′	Cos	Ctn	Tan	Sin	′

56°

34°

′	Sin	Tan	Ctn	Cos	′
0	55919	67451	1 4826	82904	60
1	55943	67493	1 4816	82887	59
2	55968	67536	1 4807	82871	58
3	55992	67578	1 4798	82855	57
4	56016	67620	1 4788	82839	56
5	56040	67663	1 4779	82822	55
6	56064	67705	1 4770	82806	54
7	56088	67748	1 4761	82790	53
8	56112	67790	1 4751	82773	52
9	56136	67832	1 4742	82757	51
10	56160	67875	1 4733	82741	50
11	56184	67917	1 4724	.82724	49
12	.56208	67960	1 4715	82708	48
13	56232	68002	1 4705	82692	47
14	56256	68045	1 4696	82675	46
15	56280	68088	1 4687	82659	45
16	56305	68130	1 4678	82643	44
17	56329	68173	1 4669	82626	43
18	56353	68215	1 4659	82610	42
19	56377	68258	1 4650	82593	41
20	56401	68301	1 4641	82577	40
21	56425	68343	1 4632	82561	39
22	56449	68386	1 4623	82544	38
23	56473	68429	1 4614	82528	37
24	56497	68471	1 4605	82511	36
25	56521	68514	1 4596	82495	35
26	56545	68557	1 4586	82478	34
27	56569	68600	1 4577	82462	33
28	56593	68642	1 4568	82446	32
29	56617	68685	1 4559	82429	31
30	56641	68728	1 4550	82413	30
31	56665	68771	1 4541	82396	29
32	56689	68814	1 4532	82380	28
33	56713	68857	1 4523	82363	27
34	56736	68900	1 4514	82347	26
35	56760	68942	1 4505	82330	25
36	56784	68985	1 4496	82314	24
37	56808	69028	1 4487	82297	23
38	56832	69071	1 4478	82281	22
39	56856	69114	1 4469	82264	21
40	56880	69157	1 4460	82248	20
41	56904	69200	1 4451	82231	19
42	56928	69243	1 4442	82214	18
43	56952	69286	1 4433	82198	17
44	56976	69329	1 4424	82181	16
45	57000	69372	1 4415	82165	15
46	57024	69416	1 4406	82148	14
47	57047	69459	1 4397	82132	13
48	57071	69502	1 4388	82115	12
49	57095	69545	1 4379	82098	11
50	57119	69588	1 4370	82082	10
51	57143	69631	1 4361	82065	9
52	57167	69675	1 4352	82048	8
53	57191	69718	1 4344	82032	7
54	57215	69761	1 4335	82015	6
55	57238	69804	1 4326	81999	5
56	47262	69847	1 4317	81982	4
57	57286	69891	1 4308	81965	3
58	57310	69934	1 4299	81949	2
59	57334	69977	1 4290	81932	1
60	57358	70021	1 4281	81915	0
′	Cos	Ctn	Tan	Sin	′

55

35°

′	Sin	Tan	Ctn	Cos	′
0	57358	70021	1 4281	81915	60
1	57381	70064	1 4273	81899	59
2	57405	70107	1 4264	81882	58
3	57429	70151	1 4255	81865	57
4	57453	70194	1 4246	81848	56
5	57477	70238	1 4237	81832	55
6	57501	70281	1 4229	81815	54
7	57524	70325	1 4220	81798	53
8	57548	70368	1 4211	81782	52
9	57572	70412	1 4202	81765	51
10	57596	70455	1 4193	81748	50
11	57619	70499	1 4185	81731	49
12	57643	70542	1 4176	81714	48
13	57667	70586	1 4167	81698	47
14	57691	70629	1 4158	81681	46
15	57715	70673	1 4150	81664	45
16	57738	70717	1 4141	81647	44
17	57762	70760	1 4132	81631	43
18	57786	70804	1 4124	81614	42
19	57810	70848	1 4115	81597	41
20	57833	70891	1 4106	81580	40
21	57857	70935	1 4097	81563	39
22	57881	70979	1 4089	81546	38
23	57904	71023	1 4080	81530	37
24	57928	71066	1 4071	81513	36
25	57952	71110	1 4063	81496	35
26	57976	71154	1 4054	81479	34
27	57999	71198	1 4045	81462	33
28	58023	71242	1 4037	81445	32
29	58047	71285	1 4028	81428	31
30	58070	71329	1 4019	81412	30
31	58094	71373	1 4011	81395	29
32	58118	71417	1 4002	81378	28
33	58141	71461	1 3994	81361	27
34	58165	71505	1 3985	81344	26
35	58189	71549	1 3976	81327	25
36	58212	71593	1 3968	81310	24
37	58236	71637	1 3959	81293	23
38	58260	71681	1 3951	81276	22
39	58283	71725	1 3942	81259	21
40	58307	71769	1 3934	81242	20
41	58330	71813	1 3925	81225	19
42	58354	71857	1 3916	81208	18
43	58378	71901	1 3908	81191	17
44	58401	71946	1 3899	81174	16
45	58425	71990	1 3891	81157	15
46	58449	72034	1 3882	81140	14
47	58472	72078	1 3874	81123	13
48	58496	72122	1 3865	81106	12
49	58519	72167	1 3857	81089	11
50	58543	72211	1 3848	81072	10
51	58567	72255	1 3840	81055	9
52	58590	72299	1 3831	81038	8
53	58614	72344	1 3823	81021	7
54	58637	72388	1 3814	81004	6
55	58661	72432	1 3806	80987	5
56	58684	72477	1 3798	80970	4
57	58708	72521	1 3789	80953	3
58	58731	72565	1 3781	80936	2
59	58755	72610	1 3772	80919	1
60	58779	72654	1 3764	80902	0
′	Cos	Ctn	Tan	Sin	′

54

36°

′	Sin	Tan	Ctn	Cos	′
0	58779	72654	1 3764	80902	60
1	58802	72699	1 3755	80885	59
2	58826	72743	1 3747	80867	58
3	58849	72788	1 3739	80850	57
4	58873	72832	1 3730	80833	56
5	58896	72877	1 3722	80816	55
6	58920	72921	1 3713	80799	54
7	58943	72966	1 3705	80782	53
8	58967	73010	1 3697	80765	52
9	58990	73055	1 3688	80748	51
10	59014	73100	1 3680	80730	50
11	59037	73144	1 3672	80713	49
12	59061	73189	1 3663	80696	48
13	59084	73234	1 3655	80679	47
14	59108	73278	1 3647	80662	46
15	59131	73323	1 3638	80644	45
16	59154	73368	1 3630	80627	44
17	59178	73413	1 3622	80610	43
18	59201	73457	1 3613	80593	42
19	59225	73502	1 3605	80576	41
20	59248	73547	1 3597	80558	40
21	59272	73592	1 3588	80541	39
22	59295	73637	1 3580	80524	38
23	59318	73681	1 3572	80507	37
24	59342	73726	1 3564	80489	36
25	59365	73771	1 3555	80472	35
26	59389	73816	1 3547	80455	34
27	59412	73861	1 3539	80438	33
28	59436	73906	1 3531	80420	32
29	59459	73951	1 3522	80403	31
30	59482	73996	1 3514	80386	30
31	59506	74041	1 3506	80368	29
32	59529	74086	1 3498	80351	28
33	59552	74131	1 3490	80334	27
34	59576	74176	1 3481	80316	26
35	59599	74221	1 3473	80299	25
36	59622	74267	1 3465	80282	24
37	59646	74312	1 3457	80264	23
38	59669	74357	1 3449	80247	22
39	59693	74402	1 3440	80230	21
40	59716	74447	1 3432	80212	20
41	59739	74492	1 3424	80195	19
42	59763	74538	1 3416	80178	18
43	59786	74583	1 3408	80160	17
44	59809	74628	1 3400	80143	16
45	59832	74674	1 3392	80125	15
46	59856	74719	1 3384	80108	14
47	59879	74764	1 3375	80091	13
48	59902	74810	1 3367	80073	12
49	59926	74855	1 3359	80056	11
50	59949	74900	1 3351	80038	10
51	59972	74946	1 3343	80021	9
52	59995	74991	1 3335	80003	8
53	60019	75037	1 3327	79986	7
54	60042	75082	1 3319	79968	6
55	60065	75128	1 3311	79951	5
56	60089	75173	1 3303	79934	4
57	60112	75219	1 3295	79916	3
58	60135	75264	1 3287	79899	2
59	60158	75310	1 3278	79881	1
60	60182	75355	1 3270	79864	0
′	Cos	Ctn	Tan	Sin	′

53°

37°

′	Sin	Tan	Ctn	Cos	′
0	60182	75355	1 3270	79864	60
1	60205	75401	1 3262	79846	59
2	60228	75447	1 3254	79829	58
3	60251	75492	1 3246	79811	57
4	60274	75538	1 3238	79793	56
5	60298	75584	1 3230	79776	55
6	60321	75629	1 3222	79758	54
7	60344	75675	1 3214	79741	53
8	60367	75721	1 3206	79723	52
9	60390	75767	1 3198	79706	51
10	60414	75812	1 3190	79688	50
11	60437	75858	1 3182	79671	49
12	60460	75904	1 3175	79653	48
13	60483	75950	1 3167	79635	47
14	60506	75996	1 3159	79618	46
15	60529	76042	1 3151	79600	45
16	60553	76088	1 3143	79583	44
17	60576	76134	1 3135	79565	43
18	60599	76180	1 3127	79547	42
19	60622	76226	1 3119	79530	41
20	60645	76272	1 3111	79512	40
21	60668	76318	1 3103	79494	39
22	60691	76364	1 3095	79477	38
23	60714	76410	1 3087	79459	37
24	60738	76456	1 3079	79441	36
25	60761	76502	1 3072	79424	35
26	60784	76548	1 3064	79406	34
27	60807	76594	1 3056	79388	33
28	60830	76640	1 3048	79371	32
29	60853	76686	1 3040	79353	31
30	60876	76733	1 3032	79335	30
31	60899	76779	1 3024	79318	29
32	60922	76825	1 3017	79300	28
33	60945	76871	1 3009	79282	27
34	60968	76918	1 3001	79264	26
35	60991	76964	1 2993	79247	25
36	61015	77010	1 2985	79229	24
37	61038	77057	1 2977	79211	23
38	61061	77103	1 2970	79193	22
39	61084	77149	1 2962	79176	21
40	61107	77196	1 2954	79158	20
41	61130	77242	1 2946	79140	19
42	61153	77289	1 2938	79122	18
43	61176	77335	1 2931	79105	17
44	61199	77382	1 2923	79087	16
45	61222	77428	1 2915	79069	15
46	61245	77475	1 2907	79051	14
47	61268	77521	1 2900	79033	13
48	61291	77568	1 2892	79016	12
49	61314	77615	1 2884	78998	11
50	61337	77661	1 2876	78980	10
51	61360	77708	1 2869	78962	9
52	61383	77754	1 2861	78944	8
53	61406	77801	1 2853	78926	7
54	61429	77848	1 2846	78908	6
55	61451	77895	1 2838	78891	5
56	61474	77941	1 2830	78873	4
57	61497	77988	1 2822	78855	3
58	61520	78035	1 2815	78837	2
59	61543	78082	1 2807	78819	1
60	61566	78129	1 2799	78801	0
′	Cos	Ctn	Tan	Sin	′

52°

38°

′	Sin	Tan	Ctn	Cos	′
0	.61566	.78129	1 2799	78801	60
1	61589	.78175	1 2792	.78783	59
2	61612	.78222	1 2784	.78765	58
3	61635	.78269	1 2776	.78747	57
4	.61658	.78316	1 2769	.78729	56
5	61681	.78363	1.2761	.78711	55
6	61704	.78410	1 2753	78694	54
7	61726	.78457	1 2746	.78676	53
8	.61749	.78504	1 2738	.78658	52
9	.61772	.78551	1 2731	.78640	51
10	.61795	.78598	1 2723	.78622	50
11	.61818	.78645	1 2715	.78604	49
12	.61841	.78692	1 2708	.78586	48
13	.61864	.78739	1.2700	.78568	47
14	.61887	.78786	1 2693	.78550	46
15	.61909	.78834	1 2685	.78532	45
16	.61932	.78881	1 2677	.78514	44
17	.61955	.78928	1 2670	.78496	43
18	.61978	.78975	1 2662	.78478	42
19	.62001	.79022	1.2655	.78460	41
20	.62024	.79070	1 2647	.78442	40
21	62046	.79117	1 2640	.78424	39
22	.62069	.79164	1 2632	.78405	38
23	.62092	.79212	1 2624	78387	37
24	62115	.79259	1 2617	.78369	36
25	.62138	.79306	1 2609	.78351	35
26	.62160	79354	1.2602	.78333	34
27	.62183	.79401	1 2594	.78315	33
28	62206	.79449	1 2587	.78297	32
29	.62229	.79496	1 2579	78279	31
30	62251	.79544	1 2572	78261	30
31	.62274	.79591	1.2564	.78243	29
32	.62297	79639	1 2557	78225	28
33	.62320	.79686	1 2549	.78206	27
34	.62342	.79734	1 2542	.78188	26
35	.62365	.79781	1 2534	.78170	25
36	.62388	.79829	1 2527	.78152	24
37	.62411	.79877	1 2519	.78134	23
38	.62433	.79924	1 2512	.78116	22
39	.62456	.79972	1 2504	.78098	21
40	.62479	.80020	1 2497	.78079	20
41	.62502	.80067	1 2489	.78061	19
42	.62524	.80115	1 2482	.78043	18
43	.62547	.80163	1 2475	.78025	17
44	.62570	.80211	1 2467	.78007	16
45	.62592	80258	1 2460	.77988	15
46	.62615	80306	1 2452	.77970	14
47	.62638	80354	1 2445	.77952	13
48	62660	80402	1 2437	.77934	12
49	.62683	.80450	1 2430	.77916	11
50	62706	.80498	1 2423	77897	10
51	.62728	.80546	1.2415	.77879	9
52	.62751	.80594	1 2408	.77861	8
53	.62774	.80642	1 2401	.77843	7
54	.62795	.80690	1.2393	.77824	6
55	.62819	.80738	1.2386	.77806	5
56	62842	.80786	1 2378	.77788	4
57	62864	80834	1 2371	.77769	3
58	62887	80882	1 2364	.77751	2
59	.62909	.80930	1 2356	.77733	1
60	62932	.80978	1.2349	.77715	0
′	Cos	Ctn	Tan	Sin	′

51°

39°

′	Sin	Tan	Ctn	Cos	′
0	62932	80978	1 2349	77715	60
1	62955	81027	1 2342	77696	59
2	.62977	81075	1 2334	77678	58
3	63000	81123	1 2327	77660	57
4	63022	81171	1 2320	77641	56
5	63045	81220	1 2312	77623	55
6	63068	81268	1 2305	77605	54
7	63090	81316	1 2298	77586	53
8	63113	81364	1 2290	77568	52
9	63135	81413	1 2283	.77550	51
10	63158	81461	1 2276	77531	50
11	.63180	81510	1 2268	77513	49
12	63203	81558	1 2261	77494	48
13	63225	81606	1 2254	77476	47
14	.63248	81655	1 2247	.77458	46
15	63271	81703	1 2239	77439	45
16	63293	81752	1 2232	77421	44
17	63316	81800	1 2225	77402	43
18	.63338	81849	1 2218	77384	42
19	63361	81898	1 2210	.77366	41
20	.63383	81946	1 2203	77347	40
21	63406	81995	1 2196	77329	39
22	63428	82044	1 2189	77310	38
23	63451	82092	1 2181	.77292	37
24	63473	82141	1 2174	.77273	36
25	63496	82190	1 2167	.77255	35
26	63518	82238	1 2160	77236	34
27	63540	.82287	1 2153	.77218	33
28	63563	82336	1 2145	.77199	32
29	63585	82385	1 2138	77181	31
30	63608	82434	1 2131	77162	30
31	63630	82483	1 2124	77144	29
32	63653	82531	1 2117	77125	28
33	63675	82580	1 2109	77107	27
34	63698	82629	1 2102	77088	26
35	63720	82678	1 2095	77070	25
36	63742	82727	1 2088	77051	24
37	63765	82776	1 2081	77033	23
38	63787	82825	1 2074	77014	22
39	63810	82874	1 2066	76996	21
40	63832	82923	1 2059	76977	20
41	63854	82972	1 2052	76959	19
42	63877	83022	1 2045	76940	18
43	63899	.83071	1 2038	76921	17
44	63922	83120	1 2031	76903	16
45	63944	83169	1 2024	76884	15
46	.63966	83218	1 2017	.76866	14
47	63989	.83268	1 2009	76847	13
48	64011	83317	1 2002	76828	12
49	.64033	83366	1 1995	76810	11
50	64056	83415	1 1988	.76791	10
51	64078	.83465	1 1981	.76772	9
52	64100	83514	1 1974	.76754	8
53	64123	.83564	1 1967	.76735	7
54	64145	.83613	1.1960	.76717	6
55	.64167	83662	1 1953	.76698	5
56	.64190	.83712	1 1946	.76679	4
57	64212	83761	1 1939	.76661	3
58	64234	83811	1 1932	.76642	2
59	.64256	.83860	1.1925	.76623	1
60	64279	83910	1.1918	.76604	0
′	Cos	Ctn	Tan	Sin	′

50°

40°

′	Sin	Tan	Ctn	Cos	′
0	64279	83910	1 1918	76604	60
1	64301	83960	1 1910	76586	59
2	64323	84009	1 1903	76567	58
3	64346	84059	1 1896	76548	57
4	64368	84108	1 1889	76530	56
5	64390	84158	1 1882	76511	55
6	64412	84208	1 1875	76492	54
7	64435	84258	1 1868	76473	53
8	64457	84307	1 1861	76455	52
9	64479	84357	1 1854	76436	51
10	64501	84407	1 1847	76417	50
11	64524	84457	1 1840	76398	49
12	64546	84507	1 1833	76380	48
13	64568	84556	1 1826	76361	47
14	64590	84606	1 1819	76342	46
15	64612	84656	1 1812	76323	45
16	64635	84706	1 1806	76304	44
17	64657	84756	1 1799	76286	43
18	64679	84806	1 1792	76267	42
19	64701	84856	1 1785	76248	41
20	64723	84906	1 1778	76229	40
21	64746	84956	1 1771	76210	39
22	64768	85006	1 1764	76192	38
23	64790	85057	1 1757	76173	37
24	64812	85107	1 1750	76154	36
25	64834	85157	1 1743	76135	35
26	64856	85207	1 1736	76116	34
27	64878	85257	1 1729	76097	33
28	64901	85308	1 1722	76078	32
29	64923	85358	1 1715	76059	31
30	64945	85408	1 1708	76041	30
31	64967	85458	1 1702	76022	29
32	64989	85509	1 1695	76003	28
33	65011	85559	1 1688	75984	27
34	65033	85609	1 1681	75965	26
35	65055	85660	1 1674	75946	25
36	65077	85710	1 1667	75927	24
37	65100	85761	1 1660	75908	23
38	65122	85811	1 1653	75889	22
39	65144	85862	1 1647	75870	21
40	65166	85912	1 1640	75851	20
41	65188	85963	1 1633	75832	19
42	65210	86014	1 1626	75813	18
43	65232	86064	1 1619	75794	17
44	65254	86115	1 1612	75775	16
45	65276	86166	1 1606	75756	15
46	65298	86216	1 1599	75738	14
47	65320	86267	1 1592	75719	13
48	65342	86318	1 1585	75700	12
49	65364	86368	1 1578	75680	11
50	65386	86419	1 1571	75661	10
51	65408	86470	1 1565	75642	9
52	65430	86521	1 1558	75623	8
53	65452	86572	1 1551	75604	7
54	65474	86623	1 1544	75585	6
55	65496	86674	1 1538	75566	5
56	65518	86725	1 1531	75547	4
57	65540	86776	1 1524	75528	3
58	65562	86827	1 1517	75509	2
59	65584	86878	1 1510	75490	1
60	65606	86929	1 1504	75471	0
′	Cos	Ctn	Tan	Sin	′

49°

41°

′	Sin	Tan	Ctn	Cos	′
0	65606	86929	1 1504	75471	60
1	65628	86980	1 1497	75452	59
2	65650	87031	1 1490	75433	58
3	65672	87082	1 1483	75414	57
4	65694	87133	1 1477	75395	56
5	65716	87184	1 1470	75375	55
6	65738	87236	1 1463	75356	54
7	65759	87287	1 1456	75337	53
8	65781	87338	1 1450	75318	52
9	65803	87389	1 1443	75299	51
10	65825	87441	1 1436	75280	50
11	65847	87492	1 1430	75261	49
12	65869	87543	1 1423	75241	48
13	65891	87595	1 1416	75222	47
14	65913	87646	1 1410	75203	46
15	65935	87698	1 1403	75184	45
16	65956	87749	1 1396	75165	44
17	65978	87801	1 1389	75146	43
18	66000	87852	1 1383	75126	42
19	66022	87904	1 1376	75107	41
20	66044	87955	1 1369	75088	40
21	66066	89007	1 1363	75069	39
22	66088	88059	1 1356	75050	38
23	66109	88110	1 1349	75030	37
24	66131	88162	1 1343	75011	36
25	66153	88214	1 1336	74992	35
26	66175	88265	1 1329	74973	34
27	66197	88317	1 1323	74953	33
28	66218	88369	1 1316	74934	32
29	66240	88421	1 1310	74915	31
30	66262	88473	1 1303	74896	30
31	66284	88524	1 1296	74876	29
32	66306	88576	1 1290	74857	28
33	66327	88628	1 1283	74838	27
34	66349	88680	1 1276	74818	26
35	66371	88732	1 1270	74799	25
36	66393	88784	1 1263	74780	24
37	66414	88836	1 1257	74760	23
38	66436	88888	1 1250	74741	22
39	66458	88940	1 1243	74722	21
40	66480	88992	1 1237	74703	20
41	66501	89045	1 1230	74683	19
42	66523	89097	1 1224	74664	18
43	66545	89149	1 1217	74644	17
44	66566	89201	1 1211	74625	16
45	66588	89253	1 1204	74606	15
46	66610	89306	1 1197	74586	14
47	66632	89358	1 1191	74567	13
48	66653	89410	1 1184	74548	12
49	66675	89463	1 1178	74528	11
50	66697	89515	1 1171	74509	10
51	66718	89567	1 1165	74489	9
52	66740	89620	1 1158	74470	8
53	66762	89672	1 1152	74451	7
54	66783	89725	1 1145	74431	6
55	66805	89777	1 1139	74412	5
56	66827	89830	1 1132	74392	4
57	66848	89883	1 1126	74373	3
58	66870	89935	1 1119	74353	2
59	66891	89988	1 1113	74334	1
60	66913	90040	1 1106	74314	0
′	Cos	Ctn	Tan	Sin	′

48°

42°

′	Sin	Tan	Ctn	Cos	′
0	66913	90040	1.1106	74314	60
1	66935	90093	1.1100	74295	59
2	66956	90146	1.1093	74276	58
3	66978	90199	1.1087	74256	57
4	66999	90251	1.1080	74237	56
5	67021	90304	1.1074	74217	55
6	67043	90357	1.1067	74198	54
7	67064	90410	1.1061	74178	53
8	67086	90463	1.1054	74159	52
9	67107	90516	1.1048	74139	51
10	67129	90569	1.1041	74120	50
11	67151	90621	1.1035	74100	49
12	67172	90674	1.1028	74080	48
13	67194	90727	1.1022	74061	47
14	67215	90781	1.1016	74041	46
15	67237	90834	1.1009	74022	45
16	67258	90887	1.1003	74002	44
17	67280	90940	1.0996	73983	43
18	67301	90993	1.0990	73963	42
19	67323	91046	1.0983	73944	41
20	67344	91099	1.0977	73924	40
21	67366	91153	1.0971	73904	39
22	67387	91206	1.0964	73885	38
23	67409	91259	1.0958	73865	37
24	67430	91313	1.0951	73846	36
25	67452	91366	1.0945	73826	35
26	67473	91419	1.0939	73806	34
27	67495	91473	1.0932	73787	33
28	67516	91526	1.0926	73767	32
29	67538	91580	1.0919	73747	31
30	67559	91633	1.0913	73728	30
31	67580	91687	1.0907	73708	29
32	67602	91740	1.0900	73688	28
33	67623	91794	1.0894	73669	27
34	67645	91847	1.0888	73649	26
35	67666	91901	1.0881	73629	25
36	67688	91955	1.0875	73610	24
37	67709	92008	1.0869	73590	23
38	67730	92062	1.0862	73570	22
39	67752	92116	1.0856	73551	21
40	67773	92170	1.0850	73531	20
41	67795	92224	1.0843	73511	19
42	67816	92277	1.0837	73491	18
43	67837	92331	1.0831	73472	17
44	67859	92385	1.0824	73452	16
45	67880	92439	1.0818	73432	15
46	67901	92493	1.0812	73413	14
47	67923	92547	1.0805	73393	13
48	67944	92601	1.0799	73373	12
49	67965	92655	1.0793	73353	11
50	67987	92709	1.0786	73333	10
51	68008	92763	1.0780	73314	9
52	68029	92817	1.0774	73294	8
53	68051	92872	1.0768	73274	7
54	68072	92926	1.0761	73254	6
55	68093	92980	1.0755	73234	5
56	68115	93034	1.0749	73215	4
57	68136	93088	1.0742	73195	3
58	68157	93143	1.0736	73175	2
59	68179	93197	1.0730	73155	1
60	68200	93252	1.0724	73135	0
′	Cos	Ctn	Tan	Sin	′

47°

43°

′	Sin	Tan	Ctn	Cos	′
0	68200	93252	1.0724	73135	60
1	68221	93306	1.0717	73116	59
2	68242	93360	1.0711	73096	58
3	68264	93415	1.0705	73076	57
4	68285	93469	1.0699	73056	56
5	68306	93524	1.0692	73036	55
6	68327	93578	1.0686	73016	54
7	68349	93633	1.0680	72996	53
8	68370	93688	1.0674	72976	52
9	68391	93742	1.0668	72957	51
10	68412	93797	1.0661	72937	50
11	68434	93852	1.0655	72917	49
12	68455	93906	1.0649	72897	48
13	68476	93961	1.0643	72877	47
14	68497	94016	1.0637	72857	46
15	68518	94071	1.0630	72837	45
16	68539	94125	1.0624	72817	44
17	68561	94180	1.0618	72797	43
18	68582	94235	1.0612	72777	42
19	68603	94290	1.0606	72757	41
20	68624	94345	1.0599	72737	40
21	68645	94400	1.0593	72717	39
22	68666	94455	1.0587	72697	38
23	68688	94510	1.0581	72677	37
24	68709	94565	1.0575	72657	36
25	68730	94620	1.0569	72637	35
26	68751	94676	1.0562	72617	34
27	68772	94731	1.0556	72597	33
28	68793	94786	1.0550	72577	32
29	68814	94841	1.0544	72557	31
30	68835	94896	1.0538	72537	30
31	68857	94952	1.0532	72517	29
32	68878	95007	1.0526	72497	28
33	68899	95062	1.0519	72477	27
34	68920	95118	1.0513	72457	26
35	68941	95173	1.0507	72437	25
36	68962	95229	1.0501	72417	24
37	68983	95284	1.0495	72397	23
38	69004	95340	1.0489	72377	22
39	69025	95395	1.0483	72357	21
40	69046	95451	1.0477	72337	20
41	69067	95506	1.0470	72317	19
42	69088	95562	1.0464	72297	18
43	69109	95618	1.0458	72277	17
44	69130	95673	1.0452	72257	16
45	69151	95729	1.0446	72236	15
46	69172	95785	1.0440	72216	14
47	69193	95841	1.0434	72196	13
48	69214	95897	1.0428	72176	12
49	69235	95952	1.0422	72156	11
50	69256	96008	1.0416	72136	10
51	69277	96064	1.0410	72116	9
52	69298	96120	1.0404	72095	8
53	69319	96176	1.0398	72075	7
54	69340	96232	1.0392	72055	6
55	69361	96288	1.0385	72035	5
56	69382	96344	1.0379	72015	4
57	69403	96400	1.0373	71995	3
58	69424	96457	1.0367	71974	2
59	69445	96513	1.0361	71954	1
60	69466	96569	1.0355	71934	0
′	Cos	Ctn	Tan	Sin	′

46°

44°

′	Sin	Tan	Ctn	Cos	′
0	69466	96569	1 0355	71934	60
1	69487	96625	1 0349	71914	59
2	69508	96681	1 0343	71894	58
3	69529	96738	1 0337	71873	57
4	69549	96794	1 0331	71853	56
5	69570	96850	1 0325	71833	55
6	69591	96907	1 0319	71813	54
7	69612	96963	1 0313	71792	53
8	69633	97020	1 0307	71772	52
9	69654	97076	1 0301	71752	51
10	69675	97133	1 0295	71732	50
11	69696	97189	1 0289	71711	49
12	69717	97246	1 0283	71691	48
13	69737	97302	1 0277	71671	47
14	69758	97359	1 0271	71650	46
15	69779	97416	1 0265	71630	45
16	69800	97472	1 0259	71610	44
17	69821	97529	1 0253	71590	43
18	69842	97586	1 0247	71569	42
19	69862	97643	1 0241	71549	41
20	69883	97700	1 0235	71529	40
21	69904	97756	1 0230	71508	39
22	69925	97813	1 0224	71488	38
23	69946	97870	1 0218	71468	37
24	69966	97927	1 0212	71447	36
25	69987	97984	1 0206	71427	35
26	70008	98041	1 0200	71407	34
27	70029	98098	1 0194	71386	33
28	70049	98155	1 0188	71366	32
29	70070	98213	1 0182	71345	31
30	70091	98270	1 0176	71325	30
31	70112	98327	1 0170	71305	29
32	70132	98384	1 0164	71284	28
33	70153	98441	1 0158	71264	27
34	70174	98499	1 0152	71243	26
35	70195	98556	1 0147	71223	25
36	70215	98613	1 0141	71203	24
37	70236	98671	1 0135	71182	23
38	70257	98728	1 0129	71162	22
39	70277	98786	1 0123	71141	21
40	70298	98843	1 0117	71121	20
41	70319	98901	1 0111	71100	19
42	70339	98958	1 0105	71080	18
43	70360	99016	1 0099	71059	17
44	70381	99073	1 0094	71039	16
45	70401	99131	1 0088	71019	15
46	70422	99189	1 0082	70998	14
47	70443	99247	1 0076	70978	13
48	70463	99304	1 0070	70957	12
49	70484	99362	1 0064	70937	11
50	70505	99420	1 0058	70916	10
51	70525	99478	1 0052	70896	9
52	70546	99536	1 0047	70875	8
53	70567	99594	1 0041	70855	7
54	70587	99652	1 0035	70834	6
55	70608	99710	1 0029	70813	5
56	70628	99768	1 0023	70793	4
57	70649	99826	1 0017	70772	3
58	70670	99884	1 0012	70752	2
59	70690	99942	1 0006	70731	1
60	70711	1 0000	1 0000	70711	0
′	Cos	Ctn	Tan	Sin	′

45°

1. $\sin\theta = \dfrac{1}{\csc\theta}$

2. $\cos\theta = \dfrac{1}{\sec\theta}$

3. $\tan\theta = \dfrac{1}{\cot\theta}$

4. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

5. $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

6. $\sin^2\theta + \cos^2\theta = 1$

7. $1 + \tan^2\theta = \sec^2\theta$

8. $1 + \cot^2\theta = \csc^2\theta$

9. $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$

10. $\cos\left(\dfrac{\pi}{2} - \theta\right) = \sin\theta$

11. $\sin 2\theta = 2\sin\theta\cos\theta$

12. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

13. $\cos 2\theta = 1 - 2\sin^2\theta$

14. $\cos 2\theta = 2\cos^2\theta - 1$

15. $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

16. $\sin^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{2}$

17. $\cos^2\dfrac{\theta}{2} = \dfrac{1 + \cos\theta}{2}$

18. $\tan^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

19. $\tan\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{\sin\theta}$

20. $\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$

21. $\cos(\theta - \varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$

22. $\sin(\theta + \varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi$

23. $\sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$

24. $\tan(\theta + \varphi) = \dfrac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi}$

25. $\tan(\theta - \varphi) = \dfrac{\tan\theta - \tan\varphi}{1 + \tan\theta\tan\varphi}$

26. $\sin\theta + \sin\varphi = 2\sin\dfrac{\theta + \varphi}{2}\cos\dfrac{\theta - \varphi}{2}$

27. $\sin\theta - \sin\varphi = 2\cos\dfrac{\theta + \varphi}{2}\sin\dfrac{\theta - \varphi}{2}$

28. $\cos\theta + \cos\varphi = 2\cos\dfrac{\theta + \varphi}{2}\cos\dfrac{\theta - \varphi}{2}$

29. $\cos\theta - \cos\varphi = -2\sin\dfrac{\theta + \varphi}{2}\sin\dfrac{\theta - \varphi}{2}$

No.	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4885	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
No.	0	1	2	3	4	5	6	7	8	9

No.	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
No.	0	1	2	3	4	5	6	7	8	9

TABLE IV. POWERS—ROOTS—RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1 000	1 000	1 0000
2	4	8	1 414	1 260	5000
3	9	27	1 732	1 442	3333
4	16	64	2 000	1 587	2500
5	25	125	2 236	1 710	2000
6	36	216	2 449	1 817	1667
7	49	343	2 646	1 913	1429
8	64	512	2 828	2 000	1250
9	81	729	3 000	2 080	1111
10	100	1 000	3 162	2 154	1000
11	121	1 331	3 317	2 224	0909
12	144	1 728	3 464	2 289	0833
13	169	2 197	3 606	2 351	0769
14	196	2 744	3 742	2 410	0714
15	225	3 375	3 873	2 466	0667
16	256	4 096	4 000	2 520	0625
17	289	4 913	4 123	2 571	0588
18	324	5 832	4 243	2 621	0556
19	361	6 859	4 359	2 668	0526
20	400	8 000	4 472	2 714	0500
21	441	9 261	4 583	2 759	0476
22	484	10 648	4 690	2 802	0455
23	529	12 167	4 796	2 844	0435
24	576	13 824	4 899	2 884	0417
25	625	15 625	5 000	2 924	0400
26	676	17 576	5 099	2 962	0385
27	729	19 683	5 196	3 000	0370
28	784	21 952	5 292	3 037	0357
29	841	24 389	5 385	3 072	0345
30	900	27 000	5 477	3 107	0333
31	961	29 791	5 568	3 141	0323
32	1 024	32 768	5 657	3 175	0312
33	1 089	35 937	5 745	3 208	0303
34	1 156	39 304	5 831	3 240	0294
35	1 225	42 875	5 916	3 271	0286
36	1 296	46 656	6 000	3 302	0278
37	1 369	50 653	6 083	3 332	0270
38	1 444	54 872	6 164	3 362	0263
39	1 521	59 319	6 245	3 391	0256
40	1 600	64 000	6 325	3 420	0250
41	1 681	68 921	6 403	3 448	0244
42	1 764	74 088	6 481	3 476	0238
43	1 849	79 507	6 557	3 503	0233
44	1 936	85 184	6 633	3 530	0227
45	2 025	91 125	6 708	3 557	0222
46	2 116	97 336	6 782	3 583	0217
47	2 209	103 823	6 856	3 609	0213
48	2 304	110 592	6 928	3 634	0208
49	2 401	117 649	7 000	3 659	0204
50	2 500	125 000	7 071	3 684	0200
51	2 601	132 651	7 141	3 708	0196
52	2 704	140 608	7 211	3 733	0192
53	2 809	148 877	7 280	3 756	0189
54	2 916	157 464	7 348	3 780	0185
55	3 025	166 375	7 416	3 803	0182
56	3 136	175 616	7 483	3 826	0179
57	3 249	185 193	7 550	3 849	0175
58	3 364	195 112	7 616	3 871	0172
59	3 481	205 379	7 681	3 893	0169
60	3 600	216 000	7 746	3 915	0167
61	3 721	226 981	7 810	3 936	0164
62	3 844	238 328	7 874	3 958	0161
63	3 969	250 047	7 937	3 979	0159
64	4 096	262 144	8 000	4 000	0156
65	4 225	274 625	8 062	4 021	0154
66	4 356	287 496	8 124	4 041	0152
67	4 489	300 763	8 185	4 062	0149
68	4 624	314 432	8 246	4 082	0147
69	4 761	328 509	8 307	4 102	0145
70	4 900	343 000	8 367	4 121	0143
71	5 041	357 911	8 426	4 141	0141
72	5 184	373 248	8 485	4 160	0139
73	5 329	389 017	8 544	4 179	0137
74	5 476	405 224	8 602	4 198	0135
75	5 625	421 875	8 660	4 217	0133
76	5 776	438 976	8 718	4 236	0132
77	5 929	456 533	8 775	4 254	0130
78	6 084	474 552	8 832	4 273	0128
79	6 241	493 039	8 888	4 291	0127
80	6 400	512 000	8 944	4 309	0125
81	6 561	531 441	9 000	4 327	0123
82	6 724	551 368	9 055	4 344	0122
83	6 889	571 787	9 110	4 362	0120
84	7 056	592 704	9 165	4 380	0119
85	7 225	614 125	9 220	4 397	0118
86	7 396	636 056	9 274	4 414	0116
87	7 569	658 503	9 327	4 431	0115
88	7 744	681 472	9 381	4 448	0114
89	7 921	704 969	9 434	4 465	0112
90	8 100	729 000	9 487	4 481	0111
91	8 281	753 571	9 539	4 498	0110
92	8 464	778 688	9 592	4 514	0109
93	8 649	804 357	9 644	4 531	0108
94	8 836	830 584	9 695	4 547	0106
95	9 025	857 375	9 747	4 563	0105
96	9 216	884 736	9 798	4 579	0104
97	9 409	912 673	9 849	4 595	0103
98	9 604	941 192	9 899	4 610	0102
99	9 801	970 299	9 950	4 626	0101
100	10 000	1 000 000	10 000	4 642	0100

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